# DISCONNECTED STABILITY WINDOWS IN SYMMETRIC–SKEW MATRIX RAYS

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Abstract. For a real symmetric matrix $D$ and a real skew-symmetric matrix $K$, we study

$$\mathcal{S}(D,K)=\{\varepsilon>0 : D+\varepsilon K \text{ is positive stable}\}.$$

This is the complementary stability condition for a matrix pencil with a pinned blocker eigenvalue. Writing $t=\varepsilon^2$, let $a_n(t)$ be the constant coefficient of the characteristic polynomial and let $\Delta_{n-1}(t)$ be its terminal nontrivial Hurwitz determinant. We prove that

$$G_n(t)=a_n(t)\Delta_{n-1}(t)$$

is positive on the strict stability set, vanishes at every positive boundary, and has degree at most $B_n=\lfloor n^2/4\rfloor$. Consequently the maximal number $c(n)$ of strict stability components satisfies $c(n)\le\lceil B_n/2\rceil$ without genericity assumptions. Exact constructions and obstructions give

$$c(1)=c(2)=c(3)=1,\quad c(4)=2,\quad c(5)=3,\quad c(6)=4,$$

with $5\le c(7)\le 6$, $6\le c(8)\le 8$, and $7\le c(9)\le 10$. Rational seeds realize two through seven components in dimensions four through nine, and a gate-and-notch construction realizes exactly $M$ components in dimension $4M-19$ for every $M\ge 7$; these patterns persist on open sets containing irreducible pairs. In the opposite direction, the guardian degree bound is algebraically sharp in every dimension: all $B_n$ positive guardian roots can be simple and distinct while the stability set remains a single final ray, robustly and irreducibly. The paper includes exact rational and integer witnesses together with Sturm, Routh–Hurwitz, interpolation, and machine-readable certificates.

## 1. Introduction

Consider a real matrix pencil

$$L_\varepsilon=A+\varepsilon B,\qquad A^T=A>0,\qquad B^T=-B. \tag{1}$$

Suppose that $\mathbb{R}e_0$ reduces both matrices and

$$Ae_0=Re_0,\qquad Be_0=0. \tag{2}$$

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The fixed eigenvalue $R$ is a *blocker*. Write $F = e_0^{\perp}$ and set

$$C_\varepsilon = L_\varepsilon|_F - RI.$$

Define the left spectral edge

$$\alpha_-(X) := \min\{\operatorname{Re} z : z \in \operatorname{spec}(X)\}.$$

The full pencil has left spectral edge equal to the blocker value precisely when

$$\alpha_-(C_\varepsilon) \geq 0.$$

We call the corresponding closed set of nonnegative amplitudes the *plateau set*. Its strict interior is

$$\mathcal{S}(C) = \{\varepsilon > 0 : C_\varepsilon \text{ is positive stable}\}, \tag{3}$$

where positive stable means that every eigenvalue has positive real part.

The first result is a fixed-dimensional obstruction. The symmetric–skew parity forces a single guardian polynomial $G_n(t) = a_n(t)\Delta_{n-1}(t)$ of degree at most $B_n = \lfloor n^2/4 \rfloor$. It is positive on the strict stability set and vanishes at every positive boundary. Counting guardian multiplicity gives $c(n) \leq \lceil B_n/2 \rceil$ without coprimality, transversality, or genericity assumptions.

The second result is an exact low-dimensional analysis. We determine the maximal component counts in dimensions four, five, and six as two, three, and four. Weak scalar extensions then give five, six, and seven components in dimensions seven, eight, and nine. The first unresolved case is $5 \leq c(7) \leq 6$. Any transverse six-window candidate is forced into one of two saturated inertia and root budgets, and a complementary-factor criterion reduces each simple guardian root to a Hurwitz test. Exact saturated examples show that root capacity alone is not enough: all available guardian roots may be present while only the final one bounds a stable interval.

The third result is an arbitrary-component construction. A gate block creates one bounded stable window and a final stable ray, while scaled near-critical notch blocks carve disjoint instability gaps inside the bounded window. Starting from the low-dimensional seeds, this produces exactly $M$ components in dimension $4M - 19$ for every $M \geq 7$. The same component counts persist on nonempty open sets containing pairs irreducible over both $\mathbb{R}$ and $\mathbb{C}$.

Finally, we prove that the guardian bound is algebraically sharp in every dimension. Rational block pencils realize all $B_n$ positive guardian roots as simple and mutually distinct while retaining only one final stable ray. This maximal hidden-root pattern also persists on open irreducible families, and a dense integer $8 \times 8$ witness is certified exactly. Repeated loss and recovery of stability is therefore neither a numerical accident nor an artifact of exact block diagonalization, but the number of guardian roots by itself does not predict the number of windows.

For later comparison, let $c(n)$ be the largest possible number of connected components of the strict stability set among eventually positive-stable $n$-dimensional symmetric–skew rays, and put

$$B_n := \left\lfloor \frac{n^2}{4} \right\rfloor. \tag{4}$$

The results below give

$$\begin{gathered} c(1) = c(2) = c(3) = 1, \qquad c(4) = 2, \qquad c(5) = 3, \\ c(6) = 4, \qquad 5 \le c(7) \le 6, \qquad 6 \le c(8) \le 8, \qquad 7 \le c(9) \le 10. \end{gathered} \tag{5}$$

More generally,

$$c(n) \ge \begin{cases} 1, & 1 \le n \le 3, \\ 2, & n = 4, \\ 3, & n = 5, \\ 4, & n = 6, \\ 5, & n = 7, \\ 6, & n = 8, \\ 7 + \left\lfloor \dfrac{n-9}{4} \right\rfloor, & n \ge 9, \end{cases} \qquad c(n) \le \begin{cases} 1, & n = 1, \\ \left\lceil \dfrac{B_n}{2} \right\rceil, & n \ge 2. \end{cases} \tag{6}$$

Thus dimensions four, five, and six are settled exactly, while dimension seven is the first unresolved extremal problem. The universal component bound is quadratic, with leading term $n^2/8$, whereas the construction below gives a linear lower bound. If zero amplitude is nonmarginal and every positive boundary point is a genuine crossing, the same count applies to the corresponding closed plateau.

The construction separates two roles. Orthogonal direct sums intersect stability sets. A gate block creates one bounded stable window and a final stable ray. A tunable notch block is stable except for a bounded gap whose relative width tends to zero at a critical parameter. A rational congruence of one gate and one notch produces a five-dimensional seed with two bounded windows and a final ray. Scaled copies of the notch block carve any prescribed finite number of disjoint gaps out of the seed's first window. Figure 1 records the interval geometry.

The question belongs to several established traditions but the conclusion is not a restatement of any one of them. Guardian maps encode loss of generalized stability in parameterized families [14]; a recent Lie-algebraic formulation unifies the classical constructions and identifies the bialternate sum with the second additive compound [3]. Modern treatments of multiplicative and additive compounds are given in [2, 13]. Matrix segments, rays, and unidirectional perturbations are central objects in parametric robust control [1, 7, 4], and symmetric or skew matrix pencils have a developed canonical-form theory [15]. Alternating stability ranges also occur in gyroscopic and nonconservative systems [11], in delay equations [5, 6], and—in a different

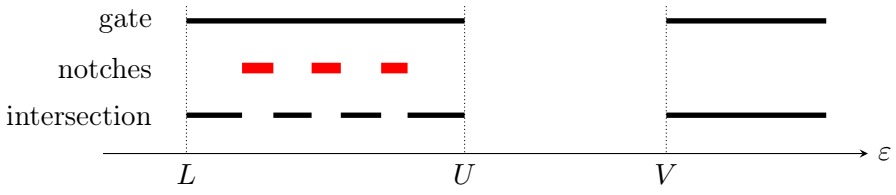


FIGURE 1. The gate contributes $(L, U) \cup (V, \infty)$. Scaled notch blocks mark disjoint deleted intervals inside $(L, U)$; their intersection with the gate leaves several bounded stability windows and the unchanged final ray.

stochastic switching-rate problem—with arbitrarily many phase transitions [12]. The contribution here is narrower: within the real symmetric–skew ray class, we give an arbitrary-component theorem, exact low-dimensional maxima, open-set persistence, irreducible realizations, and certified examples separating guardian-root complexity from window complexity.

## 2. Semialgebraic finiteness in fixed dimension

Let $D = D^T \in \mathbb{R}^{n\times n}$ and $K^T = -K$. Put

$$C_\varepsilon = D + \varepsilon K, \qquad t = \varepsilon^2,$$

and define

$$q_t(s) = \det(sI + C_\varepsilon) = s^n + a_1(t)s^{n-1} + \cdots + a_n(t). \tag{7}$$

Because

$$\det(sI + D - \varepsilon K) = \det\big((sI + D + \varepsilon K)^T\big),$$

$q_t$ depends polynomially on $t$.

Let $\Delta_k(t)$ denote the $k$th Hurwitz determinant of $q_t$, put $\Delta_0 = 1$, and use $a_0 = 1$ and $a_j = 0$ outside $0 \le j \le n$.

**Proposition 2.1** (parity degrees and semialgebraicity)**.** *For every real symmetric–skew pencil $D + \varepsilon K$:*

(1) *$a_j(t)$ is a polynomial of degree at most $\lfloor j/2 \rfloor$;*
(2)

$$\deg_t \Delta_k \le \left\lfloor \frac{k^2}{4} \right\rfloor; \tag{8}$$

(3) *the strict positive-stability set and the closed plateau set are semialgebraic subsets of the amplitude line.*

*Proof.* The coefficient $a_j$ is the sum of the principal $j \times j$ minors of $D + \varepsilon K$. Each such determinant is even in $\varepsilon$ and has degree at most $j$, proving the first claim.

Write the $k \times k$ Hurwitz matrix in the convention $H_{ij} = a_{2j-i}$, with $a_0 = 1$ and $a_r = 0$ outside the coefficient range. A nonzero determinant monomial chooses indices

$$r_j = 2j - \pi(j), \qquad 1 \le j \le k,$$

for a permutation $\pi$. Hence

$$\sum_{j=1}^{k} r_j = \frac{k(k+1)}{2}.$$

Moreover $r_j$ has the parity of $\pi(j)$, so exactly $\lceil k/2 \rceil$ selected indices are odd. Therefore every determinant monomial has $t$-degree at most

$$\sum_{j=1}^{k} \left\lfloor \frac{r_j}{2} \right\rfloor = \frac{1}{2}\left( \frac{k(k+1)}{2} - \left\lceil \frac{k}{2} \right\rceil \right) = \left\lfloor \frac{k^2}{4} \right\rfloor.$$

This proves (8).

For semialgebraicity, write $z = x + iy$. Because the roots of $q_t$ are the negatives of the eigenvalues of $D + \sqrt{t}\,K$, the complement of the closed plateau set is the projection onto the $t$-axis of

$$\{(t, x, y) : t \ge 0,\ x > 0,\ \operatorname{Re} q_t(x + iy) = 0,\ \Im q_t(x + iy) = 0\}.$$

Likewise, within $t > 0$, the complement of the strict stability set is the projection of the same algebraic equations with $x \ge 0$. Both projections are semialgebraic by the Tarski–Seidenberg theorem. Substituting $t = \varepsilon^2$ proves the assertion for amplitudes. □

The terminal Hurwitz determinant gives a considerably sharper certificate than multiplying a full Routh–Hurwitz or Liénard–Chipart list.

**Proposition 2.2** (universal quadratic guardian bound)**.** *Let*

$$G_n(t) := a_n(t)\Delta_{n-1}(t). \tag{9}$$

*If the strict positive-stability set is nonempty, then $G_n$ is a nonzero polynomial satisfying*

$$\deg_t G_n \le B_n = \left\lfloor \frac{n^2}{4} \right\rfloor, \tag{10}$$

*and every positive boundary amplitude is a positive zero of $G_n$. Consequently, for $n \ge 2$, the strict positive-stability set is a finite union of at most*

$$\left\lceil \frac{B_n}{2} \right\rceil \tag{11}$$

*open intervals; for $n = 1$ the bound is one. If, in addition, the pencil is eventually positive stable, $\alpha_-(D) \ne 0$, and every positive point of the plateau boundary is a genuine crossing rather than a tangency or isolated marginal point, then the closed plateau satisfies the same component bound.*

*Proof.* At every strictly stable parameter, the polynomial $q_t$ is Hurwitz, so $a_n(t) > 0$ and $\Delta_{n-1}(t) > 0$. Hence nonemptiness of the strict set implies that $G_n$ is not identically zero.

Let $t_0 > 0$ be a boundary point of the strict stability set. Approaching $t_0$ from the stable side shows that every root of $q_{t_0}$ lies in the closed left half-plane and at least one lies on the imaginary axis. If $s = 0$ is a root, then $a_n(t_0) = 0$. Otherwise a nonzero root $i\omega$ is accompanied by $-i\omega$, because $q_{t_0}$ has real coefficients. Orlando's formula [8] gives

$$\Delta_{n-1}(t_0) = \pm \prod_{i<j} (s_i + s_j) = 0,$$

where $s_1, \dots, s_n$ are the roots of $q_{t_0}$. Thus every positive boundary point is a zero of $G_n$.

Proposition 2.1 gives

$$\deg_t G_n \le \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{(n-1)^2}{4} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor = B_n.$$

Therefore there are at most $B_n$ distinct positive boundary points. For the sharper component count, first observe that if $D \ge 0$, then Lemma 3.3 makes the strict stability set either empty or all of $(0, \infty)$. Hence a pencil with $M \ge 2$ components has indefinite $D$ and is unstable for all sufficiently small positive $t$. Its first stability component therefore has a positive left boundary, consuming at least one degree of $G_n$.

Order the remaining components from left to right. Each gap between two adjacent components consumes at least two further units of degree. Indeed, a gap with nonempty interior has two distinct boundary zeros. If the gap consists of a single marginal point $t_0$, then $G_n(t_0) = 0$ while $G_n > 0$ on both neighboring stability intervals, so $t_0$ is a zero of even multiplicity. Thus $M \ge 2$ strict components force $\deg G_n \ge 1 + 2(M-1) = 2M - 1$. Since $\deg G_n \le B_n$, this proves (11); the one-component case is immediate.

Every positive point of the closed plateau boundary is also a zero of $G_n$: at such a point all roots lie in the closed left half-plane and at least one lies on the axis, so the preceding Orlando argument applies. Under the final hypotheses there are no isolated marginal components, including at the origin because $\alpha_-(D) \neq 0$. If $\alpha_-(D) > 0$, then $D > 0$ and the plateau is all of $[0, \infty)$, so there is one component. If there are $M \ge 2$ components, then $\alpha_-(D) < 0$ and the first plateau component has a positive left endpoint. That endpoint accounts for one positive guardian zero, and each subsequent separator accounts for at least two more. Hence $2M - 1 \le B_n$, which gives the same bound (11) for the closed plateau. □

**Lemma 2.3** (neutral-factor invariance)**.** *Let*

$$g(s) = s^m + b_1 s^{m-1} + \cdots + b_m$$

*be monic, and write $\delta_k(g)$ for its kth Hurwitz determinant. For any real $\gamma$,*

$$\Delta_k(sg) = \delta_k(g), \qquad 1 \le k \le m, \tag{12}$$

$$\Delta_k((s^2+\gamma)g) = \delta_k(g), \qquad 1 \le k \le m. \tag{13}$$

*In particular, adjoining a zero root or an opposite pair does not alter any Hurwitz minor belonging to the complementary factor.*

*Proof.* The coefficients of $sg$ through index $m$ are exactly $1, b_1, \dots, b_m$, which proves (12). For $q = (s^2+\gamma)g$, put $b_0 = 1$ and $b_j = 0$ outside $0 \le j \le m$. Its coefficients satisfy

$$a_j = b_j + \gamma b_{j-2}.$$

In the convention $H_{ij} = a_{2j-i}$, the $j$th column of the leading Hurwitz matrix of $q$ is the $j$th column for $g$ plus $\gamma$ times the preceding column. Thus

$$H_k(q) = H_k(g) R_k(\gamma),$$

where $R_k(\gamma)$ is upper bidiagonal with unit diagonal. Since $\det R_k(\gamma) = 1$, the leading determinants agree. □

**Corollary 2.4** (cofactor test for a simple guardian root)**.** *Let $t_0 > 0$ be a simple root of exactly one guardian channel.*

1. *If $a_n(t_0) = 0$ and $\Delta_{n-1}(t_0) \neq 0$, then $q_{t_0}(s) = sg(s)$ with $g$ monic of degree $n-1$. The root $t_0$ is a transverse strict-stability boundary if and only if $g$ is Hurwitz, equivalently*
$$\Delta_1(t_0), \dots, \Delta_{n-1}(t_0) > 0.$$
2. *If $\Delta_{n-1}(t_0) = 0$, $a_n(t_0) \neq 0$, and the vanishing pair is $\pm i\omega$ with $\omega > 0$, then $q_{t_0}(s) = (s^2+\omega^2)g(s)$ with $g$ monic of degree $n-2$. The root $t_0$ is a transverse strict-stability boundary if and only if $g$ is Hurwitz, equivalently*
$$\Delta_1(t_0), \dots, \Delta_{n-2}(t_0) > 0.$$

*We call such a root* boundary-admissible.

*Proof.* At the indicated root the only marginal factor is respectively $s$ or $s^2+\omega^2$. Lemma 2.3 identifies every remaining Hurwitz minor with the corresponding minor of $g$. The Routh–Hurwitz criterion therefore makes the displayed inequalities equivalent to Hurwitzness of the complementary factor. Simplicity of the guardian root then gives the transverse crossing, while a non-Hurwitz complementary factor leaves an unstable eigenvalue on both sides. □

**Proposition 2.5** (the affine second-Hurwitz barrier)**.** *For $C_\varepsilon = D + \varepsilon K$, put $t = \varepsilon^2$ and let $\Delta_2(t)$ be the second Hurwitz determinant of $\det(sI + C_\varepsilon)$. Then*

$$\Delta_2(t) = \Delta_2(0) + t\ \mathrm{tr}\big(D(-K^2)\big). \tag{14}$$

*If the nonzero skew spectrum consists of frequency planes $E_j$ with frequencies $\kappa_j > 0$, and*

$$c_j = \frac{1}{2}\operatorname{tr}(D|_{E_j})$$

*is the first real correction of that pair, then*

$$\operatorname{tr}\big(D(-K^2)\big) = 2\sum_j \kappa_j^2 c_j. \tag{15}$$

*Consequently, whenever all nonzero-pair corrections are positive, $\Delta_2$ is strictly increasing and has at most one nonnegative zero. Every boundary-admissible guardian root lies in the half-line $\{\Delta_2 > 0\}$. Thus, in the transverse seven-dimensional six-window filter, the $q = 2$ branch needs only the first guardian root to lie beyond the $\Delta_2$ threshold, while in the $q = 3$ branch the same is required of the second root; positivity then persists at all later guardian roots.*

*Proof.* Write $\tau = \operatorname{tr} D$. Newton's identities and the symmetric–skew trace cancellations give

$$a_1 = \tau, \qquad a_2(t) = e_2(D) - \frac{t}{2}\operatorname{tr} K^2,$$

and

$$a_3(t) = e_3(D) + t\left(-\frac{\tau}{2}\operatorname{tr} K^2 + \operatorname{tr}(DK^2)\right).$$

Since $\Delta_2 = a_1 a_2 - a_3$, the terms containing $\tau \operatorname{tr} K^2$ cancel and yield (14). On a real canonical frequency plane, $-K^2 = \kappa_j^2 I$, so its contribution to the trace is $\kappa_j^2 \operatorname{tr}(D|_{E_j}) = 2\kappa_j^2 c_j$; the skew kernel contributes zero. This proves (15). The final statements follow from strict monotonicity and Corollary 2.4. □

For a real symmetric matrix $S$, write

$$\iota(S) := \nu_0(S) + \min\{\nu_+(S), \nu_-(S)\},$$

where $\nu_+$, $\nu_-$, and $\nu_0$ are its positive, negative, and zero inertias. This is the largest possible dimension of a totally isotropic subspace for the bilinear form $x^T S y$.

**Proposition 2.6** (inertia bound for a determinant channel)**.** *Let $S^T = S$ and $T^T = -T$, and suppose $\det(S + \varepsilon T)$ is not identically zero. The polynomial is even in $\varepsilon$. The sum of the algebraic multiplicities of the positive roots of*

$$p(t) = \det(S + \sqrt{t}\, T)$$

*is at most $\iota(S)$. In particular, if $S$ is nonsingular with inertia $(p, q)$, the total positive-root multiplicity is at most $\min\{p, q\}$.*

*Proof.* Evenness follows from

$$\det(S + \varepsilon T) = \det((S + \varepsilon T)^T) = \det(S - \varepsilon T).$$

Consider the regular pencil $T - \lambda S$. A positive singular parameter $\varepsilon$ for $S + \varepsilon T$ corresponds, under the Möbius change $\lambda = -1/\varepsilon$, to a negative real eigenvalue of this pencil, with algebraic multiplicity preserved.

Let $x_0, \dots, x_k$ and $y_0, \dots, y_\ell$ be Jordan chains for finite pencil eigenvalues $\lambda$ and $\mu$:

$$Tx_j = \lambda S x_j + S x_{j-1}, \qquad Ty_j = \mu S y_j + S y_{j-1},$$

where $x_{-1} = y_{-1} = 0$. Skew-symmetry of $T$ gives

$$(\lambda + \mu) x_j^T S y_k = -x_{j-1}^T S y_k - x_j^T S y_{k-1}.$$

Induction on $j + k$ shows that the two root subspaces are $S$-orthogonal whenever $\lambda + \mu \neq 0$. In particular, the direct sum of all root subspaces belonging to negative real eigenvalues is totally $S$-isotropic. Its dimension is the sum of their algebraic multiplicities, and no totally isotropic subspace has dimension exceeding $\nu_0(S) + \min\{\nu_+(S), \nu_-(S)\}$. The change from $\varepsilon$ to $t = \varepsilon^2$ preserves the multiplicity of every positive root, proving the claim. □

**Lemma 2.7** (semidefinite asymptotic zero cluster)**.** *Let $S^T = S$ and $T^T = -T$. If $S + \varepsilon T$ is positive stable for all sufficiently large positive $\varepsilon$, then the compression of $S$ to $N = \ker T$ is positive semidefinite.*

*Proof.* Put $\delta = \varepsilon^{-1}$ and consider the zero spectral cluster of the normal matrix $T + \delta S$. Since zero is a semisimple eigenvalue of the skew-symmetric matrix $T$, degenerate analytic perturbation theory gives

$$\mu_j(\delta) = \delta \sigma_j + O(\delta^2),$$

where $\sigma_j$ range over the eigenvalues of the symmetric compression $S|_N$. The corresponding bounded eigenvalues of $S + \varepsilon T$ are $\varepsilon \mu_j(\varepsilon^{-1}) = \sigma_j + O(\varepsilon^{-1})$. A negative $\sigma_j$ would therefore contradict eventual positive stability. Thus $S|_N \geq 0$. □

**Proposition 2.8** (rank-sensitive zero-cluster refinement)**.** *Let $S^T = S$ and $T^T = -T$ be $d \times d$, suppose $\det(S + \varepsilon T) \not\equiv 0$, and put*

$$N = \ker T, \qquad k = \dim N.$$

*Assume that $S|_N$ is positive semidefinite of rank $h$. If*

$$p(t) = \det(S + \sqrt{t}\, T),$$

*then*

$$\deg_t p \leq \frac{d-k}{2} - \left\lceil \frac{k-h}{2} \right\rceil, \tag{16}$$

$$\sum_{\tau > 0} \operatorname{mult}_\tau p \leq \nu_0(S) + \min\{\nu_-(S), \nu_+(S) - h\}. \tag{17}$$

*Proof.* For the root count, let $L_-$ be the sum of the generalized root spaces of the regular pencil $T - \lambda S$ belonging to negative real eigenvalues. The Jordan-chain identity from Proposition 2.6 shows that $L_-$ is totally $S$-isotropic and $S$-orthogonal to $N$. Choose an $h$-dimensional subspace $P \subset N$ on which $S$ is positive definite. Then $L_- \subset P^{\perp_S}$, whose inertia is

$$(\nu_+(S) - h, \nu_-(S), \nu_0(S)).$$

The maximal isotropic dimension in that complement gives (17).

For the degree estimate, split orthogonally $\mathbb{R}^d = N \oplus N^\perp$. In this basis

$$T = 0_N \oplus T_1, \qquad T_1 \text{ invertible}, \qquad S = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where $A \geq 0$ has rank $h$. With $x = \varepsilon^{-1}$, Schur complementation gives

$$\det(S + \varepsilon T) = \varepsilon^{d-k} \det(T_1 + xC) \det\bigl(A - xB(T_1 + xC)^{-1}B^T\bigr).$$

After splitting off the positive part of $A$, the remaining $(k-h) \times (k-h)$ analytic factor has zero constant term and every entry is $O(x)$. Its determinant therefore vanishes to order at least $k - h$. Because $\det(S + \varepsilon T)$ is even in $\varepsilon$, that order is even, so it is at least $2\lceil (k-h)/2 \rceil$. This proves (16). □

Let $X^{[2]}$ denote the second additive compound of a matrix $X$, acting on $\bigwedge^2 \mathbb{R}^n$. It is linear in $X$, symmetric when $X$ is symmetric, and skew-symmetric when $X$ is skew-symmetric. Orlando's identity may be written

$$\Delta_{n-1}(t) = \det\bigl((D + \sqrt{t}\, K)^{[2]}\bigr) = \det\bigl(D^{[2]} + \sqrt{t}\, K^{[2]}\bigr),$$

with the sign fixed by positivity when $D > 0$.

**Lemma 2.9** (compound zero cluster)**.** *Suppose $D + \varepsilon K$ is eventually positive stable. Then*

$$D|_{\ker K} \geq 0, \qquad D^{[2]}|_{\ker K^{[2]}} \geq 0.$$

*If $z = \dim \ker K$ and the distinct positive frequencies of $K$ have complex multiplicities $m_1, \ldots, m_\ell$, then*

$$\dim \ker K^{[2]} = \binom{z}{2} + \sum_{j=1}^{\ell} m_j^2 \geq \left\lfloor \frac{n}{2} \right\rfloor. \tag{18}$$

*Proof.* The first assertion is Lemma 2.7. The spectrum of an additive compound consists of pairwise sums of eigenvalues. Hence positive stability of $D + \varepsilon K$ implies positive stability of $D^{[2]} + \varepsilon K^{[2]}$, and the same lemma gives the second assertion.

Complexify $K$. Its eigenvalues are zero with multiplicity $z$ and $\pm i\kappa_j$, each with multiplicity $m_j$. A zero eigenvalue of $K^{[2]}$ arises either from two zero modes or from one $+i\kappa_j$ mode and one $-i\kappa_j$ mode. This proves (18). If $r = \sum_j m_j$, then $n = z + 2r$, $\sum_j m_j^2 \geq r$, and $\binom{z}{2} \geq \lfloor z/2 \rfloor$, yielding the final inequality. □

For a simple skew spectrum, put $z = n \bmod 2$ and $m = \lfloor n/2 \rfloor$. Then $\dim \ker K = z$ and $\dim \ker K^{[2]} = m$. Let

$$h_0 = \operatorname{rank}(D|_{\ker K}), \qquad h_2 = \operatorname{rank}(D^{[2]}|_{\ker K^{[2]}}).$$

In the frequency-plane basis, the latter compression is diagonal with entries $2c_j$, where

$$\lambda_j^{\pm}(D + \varepsilon K) = \pm i\varepsilon\kappa_j + c_j + O(\varepsilon^{-1}).$$

Thus every $c_j \geq 0$, but some may vanish.

**Corollary 2.10** (asymptotic-rank refinement)**.** *Assume the preceding simple-spectrum hypotheses, and suppose $D$ and $D^{[2]}$ are nonsingular. Put*

$$q = \nu_-(D), \qquad r = \nu_-(D^{[2]}), \qquad N_2 = \binom{n}{2},$$

*and let $A, H$ be the total positive-root multiplicities of $a_n$ and $\Delta_{n-1}$. Then*

$$A \le \min\left\{ \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{z-h_0}{2} \right\rceil, q,\ n-q-h_0 \right\}, \tag{19}$$

$$H \le \min\left\{ \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - \left\lceil \frac{m-h_2}{2} \right\rceil, r,\ N_2-r-h_2 \right\}. \tag{20}$$

*Consequently, a strict stability set with $M \ge 2$ components satisfies $2M-1 \le A+H$ with the right sides bounded above by* (19)–(20)*.*

*Proof.* Apply Proposition 2.8 first to $(D, K)$ and then to $(D^{[2]}, K^{[2]})$, intersecting the resulting degree estimates with the parity degree bounds. The component assertion is the guardian multiplicity count from Proposition 2.2. □

**Remark 2.11** (strict positivity cannot be inferred)**.** The semidefinite conclusion in Lemma 2.7 is sharp. For

$$S_0 = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad T_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

one has $S_0|_{\ker T_0} = 0$, while

$$\det(sI + S_0 + \sqrt{t}\, T_0) = s^3 + 2s^2 + (t-4)s + 1$$

is Hurwitz for $t > 9/2$. The same phenomenon occurs in the terminal compound channel. Let

$$D_* = \begin{pmatrix} 3 & 0 & 5 & 0 \\ 0 & -3 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad K_* = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

Then

$$\det(sI + D_* + \sqrt{t}\, K_*) = s^4 + 2s^3 + (5t-33)s^2 + (2t+32)s + (t-6)(4t-11),$$

with $\Delta_2 = 8t - 98$ and $\Delta_3 = 200(t-17)$, so the ray is positive stable for $t > 17$. Nevertheless, in the natural basis of $\ker K_*^{[2]}$,

$$D_*^{[2]}|_{\ker K_*^{[2]}} = \operatorname{diag}(0, 2).$$

Thus a zero first correction may be stabilized at higher order, and replacing $h$ by $k$ in Proposition 2.8 is invalid without an additional nondegeneracy assumption.

**Corollary 2.12** (inertia-sensitive guardian bound)**.** *Let the strict stability set of $D + \sqrt{t}\,K$ have $M \geq 2$ components. Put*

$$R(D) := \min\left\{\left\lfloor \frac{n}{2} \right\rfloor, \iota(D)\right\} + \min\left\{\left\lfloor \frac{(n-1)^2}{4} \right\rfloor, \iota(D^{[2]})\right\}.$$

*Then*

$$2M - 1 \leq R(D), \qquad M \leq \left\lceil \frac{R(D)}{2} \right\rceil.$$

*Thus the universal quadratic guardian bound can be sharpened whenever either $D$ or its second additive compound has small isotropic capacity.*

*Proof.* Proposition 2.6, together with the parity degree bounds, shows that the total positive-root multiplicity of $a_n$ is at most the first term in $R(D)$, while that of $\Delta_{n-1}$ is at most the second. The proof of Proposition 2.2 shows that the first stability entrance consumes at least one guardian multiplicity and every later separator at least two. Hence $2M - 1 \leq R(D)$. □

**Lemma 2.13** (index-one determinant monotonicity)**.** *Suppose $D$ is nonsingular with $n-1$ positive eigenvalues and one negative eigenvalue, and let $K^T = -K$. Then*

$$a_n(t) = \det(D + \sqrt{t}\,K)$$

*has at most one positive zero. If such a zero exists, it is simple.*

*Proof.* Congruence preserves determinant zeros and inertia, so after a fixed real congruence and multiplication by a nonzero constant it is enough to take

$$J = \operatorname{diag}(-1, I_{n-1}), \qquad \widetilde{K} = \begin{pmatrix} 0 & -b^T \\ b & H \end{pmatrix}, \qquad H^T = -H.$$

For real $\varepsilon$, the matrix $I + \varepsilon H$ is invertible and has positive determinant. Its Schur complement gives

$$\begin{aligned} \det(J + \varepsilon\widetilde{K}) &= \det(I + \varepsilon H)\left(-1 + \varepsilon^2 b^T (I + \varepsilon H)^{-1} b\right) \\ &= \det(I + \varepsilon H)\left(-1 + t\, b^T (I - tH^2)^{-1} b\right), \qquad t = \varepsilon^2. \end{aligned} \tag{21}$$

The second identity follows by averaging the scalar expression at $\varepsilon$ and $-\varepsilon$. Put $P = -H^2 \geq 0$ and

$$\phi(t) = t\, b^T (I + tP)^{-1} b.$$

Then

$$\phi'(t) = b^T (I + tP)^{-2} b \geq 0,$$

with strict inequality for every $t \geq 0$ unless $b = 0$. Thus the second factor in (21) has at most one zero, and its derivative is strictly positive at any zero. Since the first factor never vanishes on the real axis, the same conclusion holds for $a_n(t)$. □

**Remark 2.14** (Bézout decomposition of the same count)**.** The number $B_n$ also has a two-channel interpretation. For $y = \omega^2$, write uniquely

$$q_t(i\omega) = E(t, y) + i\omega O(t, y).$$

If $n = 2m$, then $\deg E \le m$ and $\deg O \le m - 1$ in total degree; if $n = 2m + 1$, both total degrees are at most $m$. When $E$ and $O$ are coprime, Bézout therefore bounds nonzero imaginary-pair amplitudes by $m(m - 1)$ in even dimension and by $m^2$ in odd dimension. Zero-eigenvalue amplitudes contribute at most $m$ further roots of $a_n$. The totals are respectively $m^2$ and $m(m + 1)$, again $B_n$. Proposition 2.2 is stronger because the guardian argument needs neither coprimality nor genericity.

**Remark 2.15** (complexity gap)**.** Theorem 11.1 below gives $M$ plateau components in complementary dimension $4M - 19$ for $M \ge 7$, after seed dimensions four, five, six, seven, eight, and nine for two, three, four, five, six, and seven components. Thus the component bound is sharp through dimension six, while Corollary 11.2 gives a stronger linear lower bound in every dimension and the universal upper bound remains quadratic with leading term $n^2/8$. The first unresolved extremal value is still $c(7)$, now narrowed to $\{5, 6\}$.

**Remark 2.16** (relation to guardian maps)**.** The polynomial $G_n = a_n\Delta_{n-1}$ in Proposition 2.2 is a guardian map in the sense of Saydy, Tits, and Abed [14]: it is nonzero on the positive-stability region and vanishes whenever the family reaches its boundary. The symmetric–skew parity estimate makes this guardian unexpectedly economical, with degree at most $B_n$ rather than the cubic degree obtained by multiplying an entire Hurwitz certificate.

**Proposition 2.17** (guardian-root saturation in every dimension)**.** *For every $n \ge 2$ there is an eventually positive-stable rational symmetric–skew ray for which $a_n$ has $\lfloor n/2 \rfloor$ positive simple roots, $\Delta_{n-1}$ has $\lfloor (n-1)^2/4 \rfloor$ positive simple roots, and all of these roots are mutually distinct. Hence*

$$\deg G_n = B_n \qquad \text{and} \qquad G_n \text{ has exactly } B_n \text{ distinct positive roots.}$$

*Nevertheless its strict stability set is a single final ray.*

*Proof.* Write $n = 2m + z$ with $z \in \{0, 1\}$. On mutually orthogonal planes take

$$C_i(\varepsilon) = \begin{pmatrix} -\alpha_i & \varepsilon k_i \\ -\varepsilon k_i & \beta_i \end{pmatrix}, \qquad c_i = \beta_i - \alpha_i > 0, \qquad \delta_i(t) = k_i^2 t - \alpha_i\beta_i.$$

If $z = 1$, adjoin one positive scalar block $d$. Each plane is positive stable exactly when $\delta_i(t) > 0$, so the full direct sum is stable exactly on

$$t > \max_i \frac{\alpha_i\beta_i}{k_i^2}.$$

It remains to arrange the guardian roots.

Put $u_i = \alpha_i\beta_i$, $v_i = k_i^2$. The determinant channel is

$$a_n(t) = d^z \prod_{i=1}^{m} \delta_i(t).$$

The decomposition of the second exterior power into the within-plane lines, the cross-plane tensor products, and, when $z = 1$, the scalar–plane products gives

$$\Delta_{n-1}(t) = \left(\prod_i c_i\right)\left(\prod_{i<j} P_{ij}(t)\right)\begin{cases}1, & z = 0,\\ \prod_i R_i(t), & z = 1,\end{cases} \tag{22}$$

where

$$P_{ij}(t) = (\delta_i - \delta_j)^2 + (c_i + c_j)(c_j\delta_i + c_i\delta_j), \tag{23}$$

$$R_i(t) = \delta_i(t) + dc_i + d^2. \tag{24}$$

Indeed, $P_{ij}$ is the determinant of the Kronecker sum $C_i\otimes I + I\otimes C_j$, equivalently the resultant of the two quadratic characteristic polynomials after changing one spectral variable to its negative.

Choose integers $N_i \geq 2$ recursively so rapidly increasing that, for every $i < j$,

$$(u_i - u_j)^2 > (c_i + c_j)(c_ju_i + c_iu_j), \tag{25}$$

where

$$\alpha_i = N_i^2, \qquad \beta_i = (N_i + 1)^2, \qquad c_i = 2N_i + 1, \qquad u_i = [N_i(N_i + 1)]^2.$$

Such a choice is possible because the left side grows like $N_j^8$ and the right side like $N_j^5$ when the earlier indices are fixed. Initially take $k_i = N_i(N_i + 1)$, so $v_i = u_i$ and $\delta_i = u_i(t - 1)$. Then

$$P_{ij}(t) = (t - 1)\left((u_i - u_j)^2(t - 1) + (c_i + c_j)(c_ju_i + c_iu_j)\right).$$

Condition (25) makes both roots positive and simple. In odd dimension take $d = 1$; then

$$R_i(t) = u_i(t - 1) + c_i + 1$$

also has one positive simple root. Thus every individual factor in (22) has only positive simple roots, although several factors presently share $t = 1$.

A sufficiently small generic rational perturbation of the $k_i$ preserves all these positive simple factor roots. The finitely many coincidences between roots of distinct factors, together with the equal-frequency conditions $k_i^2 = k_j^2$, are proper algebraic conditions on the $k_i$, so the perturbation may be chosen to avoid all of them. The $m$ determinant roots and the $2\binom{m}{2} + zm$ terminal roots are then positive, simple, and mutually distinct, and the skew spectrum is simple (with one simple zero mode when $z = 1$). Their counts are

$$m + m(m - 1) = m^2 = B_{2m}, \qquad m + m(m - 1) + m = m(m + 1) = B_{2m+1}.$$

The stability set remains the single final ray displayed above. □

**Remark 2.18** (algebraic sharpness versus topological sharpness)**.** Proposition 2.17 shows that neither the degree estimate nor the positive-root count behind the guardian bound can be improved in general. What remains highly nonsharp is the conversion of roots into components: all $B_n$ roots may coexist while all but one are hidden inside unstable regions.

**Corollary 2.19** (robust irreducible guardian saturation)**.** *For every $n \geq 2$ there is a nonempty Euclidean-open set*

$$\mathcal{O}_n \subset \mathrm{Sym}(n, \mathbb{R}) \times \mathrm{Skew}(n, \mathbb{R})$$

*such that every $(D, K) \in \mathcal{O}_n$ is eventually positive stable, $G_n$ has degree $B_n$ and exactly $B_n$ positive simple roots, and $\mathcal{S}(D + \varepsilon K)$ is one final ray. Hence precisely $B_n - 1$ guardian roots are hidden. Moreover, $\mathcal{O}_n$ contains a nonempty open subset consisting entirely of irreducible pairs.*

*Proof.* Take the block pencil from Proposition 2.17 with distinct skew frequencies, and let $T$ be the largest determinant threshold. Its $B_n$ positive guardian roots are simple and distinct, $T$ is the largest of them, and the pencil is unstable for $0 \leq t < T$ and stable for $t > T$.

Choose $\eta > 0$ so that every guardian root other than $T$ lies below $T - 3\eta$. On the compact interval $[0, T - 2\eta]$ the plane whose determinant threshold is $T$ has an eigenvalue with strictly negative real part; hence the whole pencil is uniformly unstable there. Choose $R > T + 2\eta$ so large that the simple skew spectrum and strictly positive first asymptotic corrections give a uniform stability margin for $t \geq R$, by the large-amplitude argument in Lemma 12.1. The pencil is also uniformly positive stable on the compact interval $[T + 2\eta, R]$.

Polynomial coefficients depend continuously on $(D, K)$. Therefore, in a sufficiently small neighborhood of the block pencil, the guardian retains its degree and one positive simple root near each of the original $B_n$ roots, with no additional roots. Uniform instability persists on $[0, T - 2\eta]$, uniform stability persists on $[T + 2\eta, R]$ and for $t \geq R$, and there is exactly one guardian root in $(T - 2\eta, T + 2\eta)$. Since every stability boundary is a guardian root, that root is the unique entrance into the final stable ray. This gives the required open set $\mathcal{O}_n$.

Finally, pairs for which $D$ has simple spectrum and every off-diagonal entry of $K$ in a $D$-eigenbasis is nonzero are dense. Such a pair is irreducible: every $D$-invariant subspace is spanned by a subset of eigenvectors, and the nonzero cross entries of $K$ exclude every nonempty proper subset. These conditions are open. Intersecting their neighborhood with $\mathcal{O}_n$ produces the asserted open irreducible subfamily. □

**Remark 2.20** (maximal hidden complexity is robust)**.** Corollary 2.19 realizes the maximal positive-root count while keeping all but the final root uniformly hidden. The conclusion survives generic perturbations, irreducibility, and open-set persistence.

We record the dictionary between simple roots of the boundary polynomials and the crossings required by the robustness theory of Section 7.

**Lemma 2.21** (boundary crossings from simple roots)**.** *Let $t_0 > 0$ lie on the boundary of the strict positive-stability set.*

1. *If $t_0$ is a simple root of $a_n$ and $\Delta_{n-1}(t_0) \neq 0$, then $a_{n-1}(t_0) \neq 0$, the pencil has a simple real eigenvalue $\lambda(t)$ with $\lambda(t_0) = 0$ and $\lambda'(t_0) = a_n'(t_0)/a_{n-1}(t_0) \neq 0$, and no other eigenvalue lies on the imaginary axis at $t_0$.*
2. *If $t_0$ is a simple root of $\Delta_{n-1}$ and $a_n(t_0) \neq 0$, then exactly one conjugate pair $\mu(t) \pm i\omega(t)$ of eigenvalues lies on the imaginary axis at $t_0$; it is simple, $\omega(t_0) \neq 0$, and $\mu'(t_0) \neq 0$.*

*In both cases the crossing is also transverse in $\varepsilon = \sqrt{t}$.*

*Proof.* Because $t_0$ is approached from the stable side, the roots of $q_{t_0}$ lie in the closed left half-plane, and by Orlando's formula [8],

$$\Delta_{n-1} = \pm \prod_{i<j} (s_i + s_j),$$

where $s_1, \dots, s_n$ are the roots of $q_t$; a vanishing pair sum with both roots in the closed left half-plane forces both roots onto the axis.

For the first part, a double root of $q_{t_0}$ at $s = 0$ would annihilate a pair sum and hence $\Delta_{n-1}(t_0)$, so $a_{n-1}(t_0) = q_{t_0}'(0) \neq 0$ and $s = 0$ is a simple root. The implicit function theorem yields a real analytic root $s(t)$ with $s'(t_0) = -a_n'(t_0)/a_{n-1}(t_0)$, and $\lambda = -s$. A nonzero imaginary-axis eigenvalue would contribute another vanishing pair sum, contradicting $\Delta_{n-1}(t_0) \neq 0$.

For the second part, $a_n(t_0) \neq 0$ excludes a zero eigenvalue, so the marginal spectrum consists of nonzero conjugate pairs. Two distinct such pairs would make two distinct pair-sum factors vanish. If the marginal pair were non-simple, a local analytic factorization of $q_t$ into its marginal factor and a complementary nonvanishing factor would show that the relevant pair sum vanishes to order at least two. In either case $\Delta_{n-1}$ would have a zero of order at least two. Hence exactly one simple pair reaches the axis; its two roots of $q_t$ sum to $-2\mu(t)$, and the product $G(t)$ of the remaining pair sums is analytic with $G(t_0) \neq 0$, so $\mu'(t_0) = \mp \Delta_{n-1}'(t_0)/(2G(t_0)) \neq 0$.

Finally $d\lambda/d\varepsilon = 2\varepsilon\, d\lambda/dt \neq 0$ for $\varepsilon > 0$. □

## 3. Direct sums and the low-dimensional obstruction

**Lemma 3.1** (intersection under orthogonal sums)**.** *If $C_\varepsilon^{(j)} = D_j + \varepsilon K_j$, then*

$$\mathcal{S}\left(\bigoplus_{j=1}^{m} C^{(j)}\right) = \bigcap_{j=1}^{m} \mathcal{S}(C^{(j)}).$$

*The same identity holds for the closed nonnegative-real-part sets.*

*Proof.* The spectrum of an orthogonal direct sum is the union of the block spectra. □

**Proposition 3.2** (no re-entry below dimension four)**.** *Let $D + \varepsilon K$ be real symmetric–skew with $n \leq 3$. If it is positive stable for all sufficiently large $|\varepsilon|$, then its strict stability set on $\varepsilon \geq 0$ is a half-line.*

*Proof.* Orthogonally diagonalize $D$ and put $t = \varepsilon^2$. For $n = 2$, every nonconstant Hurwitz expression is affine in $t$. For $n = 3$,

$$q_t(s) = s^3 + a_1 s^2 + a_2(t)s + a_3(t),$$

where $a_2, a_3$, and $a_1 a_2 - a_3$ are affine in $t$. If each condition is positive for all sufficiently large $t$, its positivity set is a half-line. The intersection is a half-line. The case $n = 1$ is immediate. □

The half-line conclusion fails first in dimension four, but only just. The following dichotomy holds in every dimension.

**Lemma 3.3** (semidefinite dichotomy)**.** *If $D$ is positive semidefinite, the strict stability set of $D + \varepsilon K$ on $\varepsilon > 0$ is either empty or all of $(0, \infty)$. In particular a disconnected stability set requires an indefinite $D$.*

*Proof.* For every complex vector $v$ and every $\varepsilon$,

$$\operatorname{Re}\langle (D + \varepsilon K)v, v\rangle = \langle Dv, v\rangle \geq 0,$$

because $\langle Kv, v\rangle$ is purely imaginary; since the numerical range contains the spectrum, no eigenvalue ever has negative real part. Suppose some $C_{\bar\varepsilon}$ with $\bar\varepsilon > 0$ has an eigenvalue on the imaginary axis. If the eigenvalue is 0, a real null vector $v$ satisfies $v^T Dv = 0$, hence $Dv = 0$ and then $Kv = 0$, so 0 is an eigenvalue of $C_\varepsilon$ for every $\varepsilon$. If the eigenvalue is $i\omega \neq 0$ with eigenvector $v$, then $\langle Dv, v\rangle = 0$, hence $Dv = 0$ and $Kv = i(\omega/\bar\varepsilon)v$, so $C_\varepsilon$ has the purely imaginary eigenvalue $i\varepsilon\omega/\bar\varepsilon$ for every $\varepsilon$. In either case no amplitude is strictly stable. Otherwise no amplitude is marginal, and every amplitude is strictly stable. □

**Proposition 3.4** (dimension four is extremal at two components)**.** *Let $n = 4$ and suppose $D + \varepsilon K$ is positive stable for all sufficiently large $|\varepsilon|$. Then its strict stability set on $\varepsilon > 0$ has at most two connected components. By Lemma 4.1 below, the maximum over four-dimensional symmetric–skew pencils is exactly two.*

*Proof.* Put $t = \varepsilon^2$. Eventual positive stability forces $a_1 = \operatorname{tr} D > 0$, the trace being the sum of the real parts of the spectrum, and the Routh–Hurwitz criterion identifies the strict stability set with

$$S = \{t > 0 : \Delta_2(t) > 0,\ \Delta_3(t) > 0,\ a_4(t) > 0\},$$

where $\Delta_2$ is affine in $t$ while $\Delta_3$ and $a_4$ have degree at most two, and all three are eventually positive. If $\Delta_3$ or $a_4$ has degree at most one, its positivity set is a ray, and $S$ is a ray minus at most one gap, with at most two components. So let both be quadratics with positive leading coefficients. Then $\{\Delta_2 > 0\}$ meets $(0, \infty)$ in a ray $J = (\tau_0, \infty)$ with $\tau_0 \geq 0$, the sets $G_3 = \{\Delta_3 \leq 0\}$ and $G_4 = \{a_4 \leq 0\}$ are each empty, a point, or a closed bounded interval, and

$S = J \setminus (G_3 \cup G_4)$. Three components force $G_3$ and $G_4$ to be nonempty, disjoint, and contained in $(\tau_0, \infty)$, with a stable interval between $\tau_0$ and the first gap.

Suppose $\tau_0 = 0$. Then $q_t$ is Hurwitz for all small $t > 0$, so the eigenvalues of $D$ are limits of eigenvalues with positive real part and $D$ is positive semidefinite. Lemma 3.3 and eventual positive stability then give $S = (0, \infty)$, contradicting the existence of any gap.

Suppose $\tau_0 > 0$, so $\Delta_2(\tau_0) = 0$. The quartic Hurwitz identity $\Delta_3 = a_3\Delta_2 - a_1^2 a_4$ gives $\Delta_3(\tau_0) = -a_1^2 a_4(\tau_0) < 0$, because $\tau_0$ lies to the left of $G_4$, where $a_4 > 0$. Hence $\Delta_3$ is negative at $\tau_0$, so $G_3$ contains $\tau_0$ in its interior, contradicting $G_3 \subset (\tau_0, \infty)$. □

## 4. A gate block

Set

$$D_G = \operatorname{diag}\left(-\frac{5}{2}, -1, \frac{15}{2}, \frac{39}{4}\right), \qquad K_G = \begin{pmatrix} 0 & -4 & -2 & -6 \\ 4 & 0 & 2 & 8 \\ 2 & -2 & 0 & 0 \\ 6 & -8 & 0 & 0 \end{pmatrix}. \tag{26}$$

For $t = \varepsilon^2$,

$$\begin{aligned} q_G(s,t) = s^4 + \frac{55}{4}s^3 + \left(124t + \frac{61}{4}\right)s^2 + \left(894t - \frac{3405}{16}\right)s \\ + 16t^2 - \frac{873}{2}t + \frac{2925}{16}. \end{aligned} \tag{27}$$

The only nonconstant Hurwitz conditions that determine the stable windows are

$$\Delta_{3,G}(t) = \frac{184834304t^2 + 73638360t - 31865925}{256}, \tag{28}$$

$$a_{4,G}(t) = \frac{256t^2 - 6984t + 2925}{16}. \tag{29}$$

Let $L^2$ be the positive root of the numerator in (28), and let $U^2 < V^2$ be the two positive roots of the numerator in (29).

**Lemma 4.1** (gate geometry)**.** *The strict stability set of $D_G + \varepsilon K_G$ is*

$$(L, U) \cup (V, \infty). \tag{30}$$

*Moreover,*

$$\frac{L}{U} > \frac{3}{4}. \tag{31}$$

*Numerically,*

$$L \approx 0.511199, \qquad U \approx 0.652265, \qquad V \approx 5.182258.$$

*Proof.* The first and second Hurwitz determinants are positive for $t \geq 0$. The third becomes positive exactly at $L^2$, and $\Delta_{4,G} = a_{4,G}\Delta_{3,G}$. Exact root isolation gives

$$\frac{1}{4} < L^2 < \frac{27}{100}, \qquad \frac{21}{50} < U^2 < \frac{4}{9}, \qquad U^2 < V^2.$$

This proves (30), and

$$\frac{L^2}{U^2} > \frac{1/4}{4/9} = \frac{9}{16}$$

proves (31). □

## 5. A tunable notch block

Let

(32)

$$D_N(\mu) = \operatorname{diag}\left(-\frac{15}{4}, -\frac{9}{4}, 2, \frac{35}{4}\right) + \mu I, \qquad K_N = \begin{pmatrix} 0 & 4 & -3 & 7 \\ -4 & 0 & -3 & -5 \\ 3 & 3 & 0 & 5 \\ -7 & 5 & -5 & 0 \end{pmatrix}.$$

The determinant coefficient of $\det(sI + D_N(\mu) + \varepsilon K_N)$ is

$$a_{4,N}(t,\mu) = \frac{1}{64}\big(16384t^2 + (8512\mu^2 + 4448\mu - 24932)t + 64\mu^4 + 304\mu^3 - 2468\mu^2 - 915\mu + 9450\big). \tag{33}$$

Its discriminant in $t$ is, up to the positive factor $1/256$,

$$P(\mu) = 4266240\mu^4 + 3487488\mu^3 - 15182176\mu^2 - 10114352\mu + 143089. \tag{34}$$

**Lemma 5.1** (arbitrarily narrow notch)**.** *The polynomial $P$ has a unique simple root*

$$\mu_* \in \left(\frac{277}{20000}, \frac{139}{10000}\right).$$

*There is $\eta > 0$ such that, for every $\mu \in (\mu_* - \eta, \mu_*)$, the strict stability set of $D_N(\mu) + \varepsilon K_N$ is*

$$(a(\mu), b(\mu)) \cup (c(\mu), \infty), \qquad 0 < a(\mu) < b(\mu) < c(\mu), \tag{35}$$

*with*

$$\frac{a(\mu)}{b(\mu)} < \frac{3}{4}, \qquad \frac{c(\mu)}{b(\mu)} \longrightarrow 1 \quad \text{as } \mu \uparrow \mu_*. \tag{36}$$

*Proof.* Sturm's theorem applied to (34) gives the stated unique simple root. At $\mu = \mu_*$, the quadratic $a_{4,N}(t, \mu_*)$ has a double positive root

$$t_* = \frac{6233 - 1112\mu_* - 2128\mu_*^2}{8192}, \qquad t_* > \frac{3}{4}. \tag{37}$$

The third Hurwitz determinant is

$$\begin{aligned}128\Delta_{3,N}(t,\mu) = {} & 8192\mu^6 + 58368\mu^5 + 544768\mu^4 t - 19328\mu^4 \\ & + 2587648\mu^3 t - 640528\mu^3 + 8532480\mu^2 t^2 - 1212224\mu^2 t \\ & - 466740\mu^2 + 20264640\mu t^2 - 10177008\mu t + 1007494\mu \\ & + 4262448t^2 - 1406173t - 117390.\end{aligned}$$

Exact Sturm isolation on the displayed rational interval for $\mu_*$ shows that $\Delta_{3,N}(\cdot,\mu_*)$ has one negative root and one positive root $a_*^2 < 2/5$, while the first two Hurwitz determinants are strictly positive for $t \geq a_*^2$. Hence

$$\frac{a_*}{\sqrt{t_*}} < \sqrt{\frac{2/5}{3/4}} = \sqrt{\frac{8}{15}} < \frac{3}{4}.$$

Because the critical root of $P$ is simple, for $\mu < \mu_*$ sufficiently close to $\mu_*$ the determinant coefficient has two simple positive roots $b(\mu)^2 < c(\mu)^2$ converging to $t_*$. All other strict Hurwitz signs persist: on compact $t$-intervals by continuity, and for large $t$ because every Hurwitz expression involved has degree at most two in $t$ with a leading coefficient that is positive at $\mu_*$ and continuous in $\mu$, and the positive root of $\Delta_{3,N}$ converges to $a_*^2$. This proves (35)–(36). □

**Lemma 5.2** (iterated notch insertion)**.** *Suppose a base pencil has a stable component $(L,U)$ and every other stable component lies in $(U,\infty)$. Let a notch block have stability set $(a,b)\cup(c,\infty)$, where*

$$\frac{a}{b} < \frac{L}{U}, \qquad \left(\frac{c}{b}\right)^m < \frac{U}{L}$$

*for some integer $m \geq 0$. Then $m$ scaled copies of the notch block can carve $m$ pairwise disjoint instability gaps out of $(L,U)$ without changing any later component. In particular, the number of stability components increases by $m$.*

*Proof.* The case $m = 0$ is empty. Otherwise choose

$$\frac{c}{b} < r < \left(\frac{U}{L}\right)^{1/m}.$$

Because $(a/b)r^{m-1} < 1$, choose $\delta > 0$ so small that

$$(1+\delta)r^m < \frac{U}{L}, \qquad (1+\delta)\frac{a}{b}r^{m-1} < 1.$$

Set

$$\lambda_j = \frac{L}{b}(1+\delta)r^{j-1}, \qquad 1 \leq j \leq m. \tag{38}$$

Then

$$\lambda_m a < L < \lambda_1 b < \lambda_1 c < \lambda_2 b < \cdots < \lambda_m b < \lambda_m c < U.$$

The $j$th scaled notch is stable on $(\lambda_j a, \lambda_j b)\cup(\lambda_j c,\infty)$. Intersecting these sets with the base stability set deletes exactly the $m$ displayed gaps from $(L,U)$ and leaves every component above $U$ unchanged. □

## 6. A rational five-dimensional three-window seed

Set

$$\mu_0 = \frac{69}{5000}, \qquad \widehat{D} = D_G \oplus D_N(\mu_0), \qquad \widehat{K} = K_G \oplus \frac{5}{3}K_N,$$

and let

$$Q_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2/3 & -196/45 & 13/6 & -1/3 & 13/8 \\ 0 & 14/5 & 0 & 0 & 0 \\ -1/2 & -28/15 & 3/40 & 7/16 & 53/48 \\ 0 & 0 & 31/10 & 0 & 0 \\ 0 & 0 & 0 & 11/8 & 0 \\ -1 & -7/3 & 13/4 & 7/8 & 3/8 \\ 0 & 0 & 0 & 0 & 11/8 \end{pmatrix}. \tag{39}$$

Define the rational symmetric–skew pair

$$D_5 = Q_5^T \widehat{D} Q_5, \qquad K_5 = Q_5^T \widehat{K} Q_5. \tag{40}$$

This is a congruence construction rather than an invariant compression.

Let $\rho_2$ and $\rho_3$ denote the unique positive roots of the second and third Hurwitz determinants of $\det(sI + D_5 + \sqrt{t}\,K_5)$. Let $\eta_1 < \eta_2 < \eta_3$ be the positive roots of its fourth Hurwitz determinant, and let $\zeta_1 < \zeta_2$ be the positive roots of its constant coefficient $a_{5,5}(t)$.

**Proposition 6.1** (dimension five is extremal at three components)**.** *The strict stability set of $D_5 + \varepsilon K_5$ is*

$$(\sqrt{\eta_1}, \sqrt{\eta_2}) \cup (\sqrt{\eta_3}, \sqrt{\zeta_1}) \cup (\sqrt{\zeta_2}, \infty). \tag{41}$$

*All seven positive roots above are simple and satisfy*

$$\begin{aligned} &\rho_2 \in (0.011563, 0.011564), && \rho_3 \in (0.056160, 0.056161), \\ &\eta_1 \in (0.059314, 0.059315), && \eta_2 \in (0.103566, 0.103567), \\ &\eta_3 \in (0.215016, 0.215017), && \zeta_1 \in (0.228880, 0.228881), \\ &\zeta_2 \in (1.170249, 1.170250). \end{aligned}$$

*If $A = \sqrt{\eta_1}$ and $B = \sqrt{\eta_2}$ are the endpoints of the first window, then*

$$\frac{A}{B} > \frac{3}{4}. \tag{42}$$

*Moreover $50I + D_5$ is positive definite, $K_5$ has one simple zero mode and two simple nonzero frequency pairs, and all three first real large-amplitude corrections are positive. Consequently*

$$c(5) = 3.$$

*Proof.* The matrix $Q_5$ has rank five, so (40) is a rational symmetric–skew pair. Exact construction of its characteristic polynomial gives

$$\deg(a_{1,5}, \dots, a_{5,5}) = (0, 1, 1, 2, 2)$$

after the leading coefficient, and

$$\deg(\Delta_{1,5},\dots,\Delta_{4,5}) = (0,1,2,4).$$

The first determinant is a positive constant. Sturm's theorem gives one positive simple root of each of $\Delta_{2,5}$ and $\Delta_{3,5}$, three positive simple roots of $\Delta_{4,5}$, and two positive simple roots of $a_{5,5}$, ordered as displayed above. Their signs are

$$\begin{aligned} \Delta_{2,5} &> 0 && \text{for } t > \rho_2, \\ \Delta_{3,5} &> 0 && \text{for } t > \rho_3, \\ \Delta_{4,5} &> 0 && \text{on } (\eta_1,\eta_2)\cup(\eta_3,\infty), \\ a_{5,5} &> 0 && \text{on } (0,\zeta_1)\cup(\zeta_2,\infty). \end{aligned}$$

Since $\Delta_{5,5} = a_{5,5}\Delta_{4,5}$, the Routh–Hurwitz criterion gives (41). The rational isolations imply

$$\frac{A^2}{B^2} = \frac{\eta_1}{\eta_2} > \frac{0.059314}{0.103567} > \frac{9}{16},$$

which proves (42). The exact entries of $50I + D_5$ are strictly diagonally dominant with positive diagonal, so the matrix is positive definite.

For the large-amplitude assertion, put $S = -K_5^2$. The nonzero squared frequencies $r_1 < r_2$ are the roots of

$$p(r) = 268738560000r^2 - 4656819199225104r + 808193989011791545,$$

with exact isolations

$$r_1 \in (175.324, 175.325), \qquad r_2 \in (17153.112, 17153.113).$$

The correction on the simple zero mode is

$$c_0 = \frac{494745751976054098751}{808193989011791545000} > 0. \tag{43}$$

If $v_j$ is a normalized $i\sqrt{r_j}$-eigenvector of $K_5$, the spectral projector of $S$ gives

$$\begin{aligned} \langle D_5 v_j, v_j\rangle &= \frac{H(r_j)}{1296000000\, r_j p'(r_j)}, \\ H(r) &= h_1 r - h_0, \\ h_1 &= 10173770870154449561092785 6, \\ h_0 &= 47948364415229429369964657505. \end{aligned} \tag{44}$$

The single root of $H$ lies in $(471.293, 471.294)$. Hence $H(r_j)$ and $p'(r_j)$ have the same signs $-,+$, proving that both nonzero-frequency corrections are positive. The exact Sturm counts, projector identity, zero-mode calculation, and diagonal-dominance check are reproduced by `verify_five_dimensional_seed.py`.

Finally, Proposition 2.2 gives $c(5) \le \lceil B_5/2\rceil = 3$, while (41) attains three. □

## 7. The exact six-dimensional extremum

The universal guardian count gives $c(6) \le 5$. Saturating that count would force an exceptionally rigid crossing pattern. The index-one determinant lemma rules out the only pattern not already excluded by the degree of the terminal Hurwitz determinant.

**Proposition 7.1** (five components are impossible in dimension six)**.** *Every eventually positive-stable six-dimensional symmetric–skew ray has at most four strict stability components.*

*Proof.* The lower bound follows by adjoining one stable scalar block to the five-dimensional seed. Suppose, for contradiction, that a six-dimensional ray has five strict stability components. The semidefinite dichotomy implies that its first component begins at a positive parameter, and eventual stability makes its fifth component an unbounded final ray. The first entrance consumes at least one unit of guardian multiplicity, and each of the four later separators consumes at least two. Since $B_6 = 9$, equality holds throughout: $G_6$ has degree nine, the first entrance has multiplicity one, and each later separator has total guardian multiplicity exactly two. There are no additional positive guardian zeros before the first component or inside an open separator. In particular $G_6(0) \neq 0$, so $D$ is nonsingular.

Let $q$ be the number of negative eigenvalues of $D$. The simple first entrance is either a zero-eigenvalue crossing, changing the unstable count by one, or a nonzero imaginary-pair crossing, changing it by two. Hence $q \in \{1, 2\}$. Each later separator uses two degrees in a single boundary channel. For a separator with nonempty interior, its exit and re-entry must be of the same type: a one-dimensional instability can only be removed by a second zero-eigenvalue crossing, while a conjugate-pair instability can only be removed by a second pair crossing. If the separator is a single marginal parameter, stability on both sides forces even-order contact with the axis; the equality case therefore gives one double zero of either $a_6$ or $\Delta_5$. Call the two possibilities a zero separator and a pair separator, respectively.

If $q = 1$, the first entrance is a zero-eigenvalue crossing. By Lemma 2.13, $a_6$ has no further positive zero. All four later separators must therefore be pair separators, consuming eight degrees of $\Delta_5$. This contradicts

$$\deg_t \Delta_5 \le \left\lfloor \frac{25}{4} \right\rfloor = 6.$$

If $q = 2$, the first entrance is a pair crossing. Let $z$ be the number of later zero separators. They consume $2z$ degrees of $a_6$, so $2z \le \deg_t a_6 \le 3$ and hence $z \le 1$. The initial pair crossing and the remaining $4 - z$ pair separators consume at least

$$1 + 2(4 - z) = 9 - 2z$$

degrees of $\Delta_5$. Its degree bound forces $9 - 2z \le 6$, hence $z \ge 2$, a contradiction. Therefore five components are impossible. □

The preceding upper bound uses only degree. The inertia refinement gives a more rigid description of any four-component extremizer.

**Proposition 7.2** (corrected asymptotic-rank filter in dimension six)**.** *Suppose a six-dimensional eventually positive-stable ray has four strict stability components, $D$ and $D^{[2]}$ are nonsingular, $K$ has three distinct nonzero frequencies, and every positive root of $a_6\Delta_5$ is simple. Write*

$$q = \nu_-(D), \qquad r = \nu_-(D^{[2]}), \qquad h = \operatorname{rank}(D^{[2]}|_{\ker K^{[2]}}).$$

*Then $1 \le h \le 3$, and only the following grouped cases are possible:*

| $h$ | $q$ | $r$ | $(A, H)$ |
|---|---|---|---|
| 3 | 2 | 5, 7 | (2, 5) |
| 3 | 2 | 6 | (2, 6) |
| 3 | 3 | 5, 7 | (3, 5) |
| 3 | 3 | 6 | (3, 6) |
| 3 | 4 | 6 | (2, 6) |
| 2 | 2 | 5, 7 | (2, 5) |
| 2 | 3 | 5, 7 | (3, 5) |
| 1 | 2 | 5, 7, 9 | (2, 5) |
| 1 | 3 | 5, 7, 9 | (3, 5) |

*Here $A$ and $H$ are the total positive-root multiplicities of $a_6$ and $\Delta_5$. In particular, $r = 8$ is impossible for every asymptotic rank. The previously listed seven $(q, r)$ strata are exactly the nondegenerate case $h = 3$; the only additional high-index possibility is the exceptional rank-one branch $r = 9$.*

*Proof.* For three distinct nonzero frequencies, $\ker K^{[2]}$ has dimension three. In the frequency-plane basis its $D^{[2]}$ compression is $\operatorname{diag}(2c_1, 2c_2, 2c_3)$ with $c_j \ge 0$. Its trace is $\operatorname{tr} D > 0$, so $1 \le h \le 3$. Corollary 2.10 and the parity degree bounds give

$$A \le \min\{q, 6 - q, 3\},$$

and

$$H \le \min\left\{6 - \left\lceil \frac{3-h}{2} \right\rceil, r,\ 15 - r - h\right\}.$$

Because all positive roots are simple and the ray is eventually stable,

$$A \equiv q \pmod 2, \qquad H \equiv r \pmod 2.$$

Before the first stable component, the unstable count must fall from $q$ to zero. A simple zero crossing changes it by at most one and a simple pair crossing by at most two. The first entrance therefore costs at least $\lceil q/2 \rceil$ roots, and the three later separators cost six more:

$$A + H \ge 6 + \left\lceil \frac{q}{2} \right\rceil.$$

As before, $q = 0, 6$ are impossible, and the pair-sum inertia obeys

$$\binom{q}{2} \le r \le \binom{q}{2} + q(6 - q).$$

Enumerating these inequalities for $h = 1, 2, 3$ gives exactly the displayed table. If $r = 8$, parity forces $H$ even, whereas the rank-sensitive degree and capacity bounds give $H \le 5$ for $h < 3$ and $H \le 4$ for $h = 3$; the required entrance-and-separator count cannot be met. The value $r = 9$ survives only when $h = 1$, where $H = 5$ saturates both the degree and inertia capacities. □

**Remark 7.3** (meaning of the corrected six-dimensional filter)**.** The integer extremizer below lies in the nondegenerate stratum

$$(h, q, r, A, H) = (3, 2, 6, 2, 6).$$

Thus Proposition 7.2 is sharp at the level of its allowed strata and root budgets. The exceptional lower-rank branches remain relevant to the structure theorem, but they are not needed for attainment of the exact maximum.

7.1. **An integer four-window extremizer.** Let

$$D_6 = 10^{-9}\mathsf{D}_6, \qquad K_6 = 10^{-9}\mathsf{K}_6,$$

where

$$\mathsf{D}_6 = \begin{pmatrix} 5142123 & 21824940 & -10949427 & -8247299 & -10125367 & 0 \\ 21824940 & 994088956 & -14421431 & -27094692 & -29607217 & 0 \\ -10949427 & -14421431 & -55552357 & 13559280 & -519009 & 0 \\ -8247299 & -27094692 & 13559280 & -4022030 & 11843595 & 0 \\ -10125367 & -29607217 & -519009 & 11843595 & 52873989 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6567014 \end{pmatrix}$$

and

$$\mathsf{K}_6 = \begin{pmatrix} 0 & 28364852 & -39228271 & -28391511 & -57516137 & -24651790 \\ -28364852 & 0 & -44525354 & -82528658 & -120068545 & -977964173 \\ 39228271 & 44525354 & 0 & 107922732 & 238297085 & 61030654 \\ 28391511 & 82528658 & -107922732 & 0 & -40642015 & 24431401 \\ 57516137 & 120068545 & -238297085 & 40642015 & 0 & 61420639 \\ 24651790 & 977964173 & -61030654 & -24431401 & -61420639 & 0 \end{pmatrix}.$$

The common factor $10^{-9}$ is included only to keep the blocker shift at the same convenient value used elsewhere; it does not change the amplitude parameters.

Let $\rho < \eta_1 < \eta_2 < \eta_3 < \eta_4 < \eta_5$ be the six positive roots of $\Delta_{5,6}(t)$, and let $\zeta_1 < \zeta_2$ be the two positive roots of $a_{6,6}(t)$.

**Proposition 7.4** (dimension six is extremal at four components)**.** *The strict stability set of $D_6 + \varepsilon K_6$ is*

$$(\sqrt{\eta_1}, \sqrt{\eta_2}) \cup (\sqrt{\eta_3}, \sqrt{\eta_4}) \cup (\sqrt{\eta_5}, \sqrt{\zeta_1}) \cup (\sqrt{\zeta_2}, \infty). \tag{45}$$

*All eight positive guardian roots are simple and satisfy*

$$\begin{aligned} \rho &\in (0.00316355, 0.00316356), & \eta_1 &\in (0.06172988, 0.06172989), \\ \eta_2 &\in (0.06504387, 0.06504388), & \eta_3 &\in (0.08179916, 0.08179917), \\ \eta_4 &\in (0.13478195, 0.13478196), & \eta_5 &\in (0.19725682, 0.19725683), \\ \zeta_1 &\in (0.21816110, 0.21816111), & \zeta_2 &\in (0.31478812, 0.31478813). \end{aligned}$$

*The first window satisfies*

$$\sqrt{\eta_1/\eta_2} > \frac{3}{4}.$$

*Moreover* $50I + D_6$ *is positive definite,* $K_6$ *has three simple nonzero frequency pairs, and all three first real large-amplitude corrections are positive. The pencil realizes*

$$(h, q, r, A, H) = (3, 2, 6, 2, 6),$$

*and consequently*

$$c(6) = 4.$$

*Proof.* The displayed matrices are exactly symmetric and skew-symmetric. Exact interpolation of

$$q_t(s) = \det(sI + \mathsf{D}_6 + \sqrt{t}\,\mathsf{K}_6)$$

gives coefficient degrees $(0, 1, 1, 2, 2, 3)$ and Hurwitz-determinant degrees $(0, 1, 2, 4, 6, 9)$. Sturm's theorem gives six positive simple roots of $\Delta_{5,6}$ and two positive simple roots of $a_{6,6}$, with the rational isolations displayed above and no other positive guardian roots. Exact Routh–Hurwitz evaluation in the nine complementary regions gives the sign pattern

$$\text{U, U, S, U, S, U, S, U, S,}$$

where S denotes strict stability. This proves (45); the root $\rho$ is a guardian zero occurring inside the initial unstable region. The same rational isolations give

$$\frac{\eta_1}{\eta_2} > \frac{0.06172988}{0.06504388} > \frac{9}{16}.$$

Exact spectral isolation gives $q = \nu_-(D_6) = 2$ and $r = \nu_-(D_6^{[2]}) = 6$. The characteristic polynomial of $\mathsf{K}_6$ has the form

$$\begin{aligned} &x^6 + 1064867901609958681x^4 \\ &\quad + 6860295015749024593865814084554501 3x^2 \\ &\quad + 346218893546524064002713275599618118213322606972 1. \end{aligned}$$

The associated cubic in the squared frequencies has three distinct positive roots, so the skew spectrum consists of three simple nonzero conjugate pairs. Eventual stability makes the three first real corrections nonnegative. Because $\deg_t \Delta_{5,6} = 6$ is the full simple-frequency degree, none of those corrections vanishes; hence all three are positive and $h = 3$. Finally, $50I + D_6$ is strictly diagonally dominant with positive diagonal. The exact Sturm counts, Routh signs, inertia calculation, frequency certificate, and diagonal-dominance check are reproduced by `verify_six_dimensional_seed.py`. Proposition 7.1 provides the matching upper bound. □

## 8. A seven-dimensional five-window seed

The six-dimensional extremizer has an additional useful feature: a weakly coupled scalar mode can create a new determinant instability gap inside its final stable ray while perturbing the inherited pair crossings only at second order. We give the resulting rational pencil explicitly.

Let $D_6, K_6$ be the matrices in Proposition 7.4, set

$$v = (-1902, -324, 3016, 4066, -8380, 653)^T,$$

and define

$$D_7 = \begin{pmatrix} D_6 & 0 \\ 0 & 9 \cdot 10^{-12} \end{pmatrix}, \qquad K_7 = \begin{pmatrix} K_6 & 10^{-10} v \\ -10^{-10} v^T & 0 \end{pmatrix}. \tag{46}$$

Thus $D_7$ and $K_7$ have the common exact scale $10^{-12}$: their upper-left blocks are $1000\mathsf{D}_6$ and $1000\mathsf{K}_6$, the last diagonal entry of the symmetric integer matrix is 9, and the last column of the skew integer matrix is $100v$.

The mechanism is visible from the Schur complement. Writing $A_6(\varepsilon) = D_6 + \varepsilon K_6$ and $c = 10^{-10} v$, one has

$$\det(D_7 + \varepsilon K_7) = \det A_6(\varepsilon) \left(9 \cdot 10^{-12} + \varepsilon^2 c^T A_6(\varepsilon)^{-1} c\right) \tag{47}$$

whenever $A_6(\varepsilon)$ is invertible. The scalar factor is positive near the beginning of the final six-dimensional stable ray, negative on an intermediate interval, and positive again at infinity.

Let $\rho < \eta_1 < \cdots < \eta_7$ be the eight positive roots of $\Delta_{6,7}(t)$, and let $\zeta_1 < \zeta_2$ be the two positive roots of $a_{7,7}(t)$.

**Proposition 8.1** (five windows in dimension seven)**.** *The strict stability set of $D_7 + \varepsilon K_7$ is*

$$(\sqrt{\eta_1}, \sqrt{\eta_2}) \cup (\sqrt{\eta_3}, \sqrt{\eta_4}) \cup (\sqrt{\eta_5}, \sqrt{\eta_6}) \cup (\sqrt{\eta_7}, \sqrt{\zeta_1}) \cup (\sqrt{\zeta_2}, \infty). \tag{48}$$

*All ten positive guardian roots are simple and satisfy*

$$\begin{aligned} \rho &\in (0.00316355, 0.00316356), & \eta_1 &\in (0.06172988, 0.06172989), \\ \eta_2 &\in (0.06504381, 0.06504382), & \eta_3 &\in (0.08180044, 0.08180045), \\ \eta_4 &\in (0.13474364, 0.13474365), & \eta_5 &\in (0.19768918, 0.19768919), \\ \eta_6 &\in (0.21770849, 0.21770850), & \eta_7 &\in (0.31482551, 0.31482552), \\ \zeta_1 &\in (0.44152207, 0.44152208), & \zeta_2 &\in (1.59202830, 1.59202831). \end{aligned}$$

*The first window satisfies $\sqrt{\eta_1/\eta_2} > 3/4$. Moreover $50I + D_7$ is positive definite, $K_7$ has one simple zero mode and three distinct nonzero frequency pairs, and all four first real large-amplitude corrections are positive. The pencil realizes*

$$(q, r, A, H) = (2, 8, 2, 8),$$

*and therefore*

$$5 \le c(7) \le 6.$$

*Proof.* Using the common integer scale $10^{-12}$, exact interpolation of $q_t(s) = \det(sI + D_7 + \sqrt{t}K_7)$ gives coefficient degrees $(0, 1, 1, 2, 2, 3, 3)$ and Hurwitz-determinant degrees $(0, 1, 2, 4, 6, 9, 12)$. Sturm's theorem gives eight positive simple roots of $\Delta_{6,7}$ and two positive simple roots of $a_{7,7}$, with the displayed rational isolations and no other positive guardian roots. Exact Routh–Hurwitz evaluation in the eleven complementary regions gives

$$\mathrm{U},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S}.$$

This proves (48). As in dimension six, $\rho$ is a guardian zero inside the initial unstable region. The first two isolations give $\eta_1/\eta_2 > 9/16$.

Exact spectral isolation gives $q = \nu_-(D_7) = 2$ and $r = \nu_-(D_7^{[2]}) = 8$. The characteristic polynomial of $K_7$ is $x$ times an even sextic whose associated cubic in the squared frequencies has three distinct positive roots. Hence the skew spectrum consists of one simple zero and three simple nonzero conjugate pairs. The full degrees $\deg_t a_{7,7} = 3$ and $\deg_t \Delta_{6,7} = 9$, together with eventual stability, exclude a zero first correction on the zero mode or on any frequency plane; all four corrections are therefore positive. Finally, $50I + D_7$ is strictly diagonally dominant with positive diagonal. The exact Schur identity (47), Sturm counts, Routh signs, inertia calculation, frequency certificate, and blocker shift are reproduced by `verify_seven_dimensional_seed.py`. The upper bound is the universal guardian estimate $c(7) \le \lceil B_7/2 \rceil = 6$. □

**Proposition 8.2** (saturation filter for six windows in dimension seven)**.** *Suppose a seven-dimensional eventually positive-stable symmetric–skew ray has six strict stability components. Assume that $D$ and $D^{[2]}$ are nonsingular, $K$ has one simple zero mode and three distinct nonzero frequency pairs, and every positive root of $a_7\Delta_6$ is simple. Put*

$$q = \nu_-(D), \qquad r = \nu_-(D^{[2]}),$$

*and let $A$ and $H$ be the total positive-root counts of $a_7$ and $\Delta_6$. Then exactly one of the following alternatives holds:*

$$(q, r, A, H) = (2, 9, 2, 9), \tag{49}$$

$$(q, r, A, H) = (3, 9, 3, 9). \tag{50}$$

*In both alternatives the compression $D^{[2]}|_{\ker K^{[2]}}$ is positive definite. In* (49)*, every positive guardian root lies on the stability boundary. In* (50)*, the first of the two entrance roots is hidden inside the initial unstable region, while the second entrance root and all ten later separator roots lie on the stability boundary. The first entrance is one conjugate-pair crossing in* (49)*, and one zero plus one conjugate-pair crossing in* (50)*. Among the five later separators, exactly one is a zero-eigenvalue separator and the other four are conjugate-pair separators.*

*Proof.* Let $h_0$ and $h_2$ be the ranks of the compressions of $D$ and $D^{[2]}$ to $\ker K$ and $\ker K^{[2]}$, respectively. Here $\dim\ker K = 1$ and $\dim\ker K^{[2]} = 3$.

Corollary 2.10 gives

$$A \le 3, \qquad H \le 9.$$

The five separators after the first stable component consume at least ten positive guardian roots. Before that component the unstable dimension must fall from $q$ to zero. A simple zero crossing changes it by at most one and a simple conjugate-pair crossing by at most two, so the first entrance consumes at least $\lceil q/2 \rceil$ roots. Consequently

$$A + H \ge 10 + \left\lceil \frac{q}{2} \right\rceil . \tag{51}$$

If $q = 1$, the index-one determinant lemma gives $A \le 1$, whence $A+H \le 10$, contradicting (51). If $q = 2$, the inertia bound gives $A \le 2$, so equality throughout forces $A = 2$, $H = 9$, and a one-root first entrance. That root cannot be a zero crossing, and hence is a conjugate-pair crossing. The terminal estimate

$$H \le \min \left\{ 9 - \left\lceil \frac{3 - h_2}{2} \right\rceil , r, 21 - r - h_2 \right\}$$

then forces $h_2 = 3$ and $r = 9$. Since the entrance and the five separators already consume all eleven positive guardian roots, there are no hidden positive roots; every root is boundary-admissible by Corollary 2.4. Each two-root separator must use the same channel twice: after leaving a stable interval a single zero crossing creates one unstable real root, while a single pair crossing creates two, and the other channel cannot return the unstable dimension to zero in one further crossing. The two determinant roots therefore form one zero separator, while the eight remaining terminal roots form four pair separators. This proves (49) and its asserted crossing pattern.

If $q = 3$, (51) and the degree bounds force $A = 3$ and $H = 9$. The rank-sensitive degree estimate for $a_7$ then gives $h_0 = 1$, while the terminal estimate again gives $h_2 = 3$ and $r = 9$. The first entrance uses exactly two roots; reducing three unstable modes to zero requires one zero crossing and one pair crossing. The first of those roots is hidden inside the initial unstable region, while the second is the left boundary of the first stable component. The remaining two determinant roots and eight terminal roots give exactly one zero separator and four pair separators. Thus all twelve positive guardian roots are used, but exactly eleven are boundary-admissible.

For $q = 4$, if $h_0 = 1$ then the root-capacity estimate gives $A \le 2$, while if $h_0 = 0$ the degree-loss estimate also gives $A \le 2$. Thus $A + H \le 11 < 12$, contradicting (51). If $q = 5$ or 6, the same estimates give $A \le 2$ and hence $A + H \le 11$, whereas (51) requires at least 13. Finally, $q = 0$ makes $D$ positive definite and the whole ray stable, while $q = 7$ contradicts the positive trace required by eventual stability. The two listed alternatives are therefore exhaustive. □

**Corollary 8.3** (finite guardian-word filter)**.** *Under the hypotheses of Proposition 8.2, write $A$ for a determinant-channel root and $H$ for a terminal-Hurwitz root, in increasing order of $t$.*

(1) *In the $q = 2$ branch the word has the form*

$$HP_1P_2P_3P_4P_5, \qquad \{P_1, \dots, P_5\} = \{AA, HH, HH, HH, HH\}.$$

*There are five possible words, and all eleven roots must be boundary-admissible.*

(2) *In the $q = 3$ branch the word has the form*

$$AHP_1P_2P_3P_4P_5 \quad \text{or} \quad HAP_1P_2P_3P_4P_5,$$
$$\{P_1, \dots, P_5\} = \{AA, HH, HH, HH, HH\}.$$

*There are ten possible words. The first root is hidden; the other eleven must be boundary-admissible.*

*Thus the transverse six-window search reduces to fifteen channel words and eleven simultaneous Hurwitz-cofactor tests.*

*Proof.* The first entrance uses one $H$ root for $q = 2$, and one root of each channel for $q = 3$. Each later separator uses two roots from the same channel by Proposition 8.2. The remaining channel counts are one $AA$ pair and four $HH$ pairs. Corollary 2.4 gives the admissibility statements. □

**Proposition 8.4** (the scalar-extension chamber)**.** *Let $D_6$ be the symmetric matrix of the exact four-window seed in Proposition 7.4, with ordered eigenvalues $\lambda_1 < \cdots < \lambda_6$, and consider a one-dimensional extension*

$$D_7(d) = \operatorname{diag}(D_6, d), \qquad K_7(c) = \begin{pmatrix} K_6 & c \\ -c^T & 0 \end{pmatrix}.$$

*Changing $c$ does not change either inertia of the symmetric channels. More precisely,*

$$\nu_-(D_7(d)) = \nu_-(D_6) + \mathbf{1}_{\{d<0\}}, \tag{52}$$

$$\nu_-(D_7(d)^{[2]}) = \nu_-(D_6^{[2]}) + \#\{j : \lambda_j + d < 0\}. \tag{53}$$

*For the certified six-dimensional seed,*

$$\nu_-(D_6) = 2, \qquad \nu_-(D_6^{[2]}) = 6,$$

*and*

$$\lambda_3 \in (0.006567013, 0.006567015), \qquad \lambda_4 \in (0.008610495, 0.008610498).$$

*Hence every positive-scalar extension has $q = 2$ and $r \le 8$, so no choice of its skew coupling $c$ can enter the saturated $q = 2$ six-window stratum $(2, 9, 2, 9)$. The only scalar-extension chamber with $(q, r) = (3, 9)$ is*

$$-\lambda_4 < d < -\lambda_3. \tag{54}$$

*In particular, the missing ninth positive root of the terminal determinant cannot be created from the displayed five-window seed by varying its weak extension vector alone: the symmetric block must also cross an inertia wall.*

*Proof.* The spectrum of $D_7(d)$ is the spectrum of $D_6$ together with $d$, which proves (52). The eigenvalues of the second additive compound of a symmetric matrix are its pairwise eigenvalue sums. The old sums contribute $\nu_-(D_6^{[2]})$, and the six new sums are $\lambda_j + d$, proving (53). Exact rational isolation of the characteristic polynomial of $D_6$ gives the displayed intervals. If $d > 0$, only the two negative eigenvalues of $D_6$ can form new negative sums, so $r \le 6 + 2 = 8$. If $d < 0$, the condition $q = 3$ is automatic, while exactly three new sums are negative precisely when $-\lambda_4 < d < -\lambda_3$. The seven-dimensional saturation filter then gives the stated no-go conclusion. □

**Proposition 8.5** (guardian saturation without window saturation)**.** *Let*

$$\widehat{D} = \operatorname{diag}(-413, -112, -92, 59, 104, 219, 2446)$$

*and*

$$\widehat{K} = \begin{pmatrix} 0 & 896 & 306 & 101 & -686 & -1812 & -987 \\ -896 & 0 & 1241 & -707 & 864 & -851 & -560 \\ -306 & -1241 & 0 & 802 & -1131 & 1671 & 1019 \\ -101 & 707 & -802 & 0 & 995 & -254 & 223 \\ 686 & -864 & 1131 & -995 & 0 & 61 & -1253 \\ 1812 & 851 & -1671 & 254 & -61 & 0 & -798 \\ 987 & 560 & -1019 & -223 & 1253 & 798 & 0 \end{pmatrix}.$$

*For the ray $\widehat{D} + \varepsilon\widehat{K}$ one has*

$$(q, r, A, H) = (3, 9, 3, 9).$$

*The skew matrix has one simple zero mode and three distinct nonzero frequency pairs; the zero-mode correction is positive; and $\widehat{D}^{[2]}|_{\ker \widehat{K}^{[2]}}$ is positive definite. If $\alpha_1 < \alpha_2 < \alpha_3$ are the positive roots of $a_7$ and $\beta_1 < \cdots < \beta_9$ are the positive roots of $\Delta_6$, then all roots are simple and*

$$\beta_1 < \beta_2 < \alpha_1 < \beta_3 < \beta_4 < \beta_5 < \beta_6 < \alpha_2 < \beta_7 < \beta_8 < \alpha_3 < \beta_9. \tag{55}$$

*Among these twelve roots, only $\beta_9$ is boundary-admissible. Nevertheless,*

$$\mathcal{S}(\widehat{D} + \varepsilon\widehat{K}) = (\sqrt{\beta_9}, \infty). \tag{56}$$

*Thus the full root and asymptotic budget in* (50) *is realizable, but does not by itself produce six stability components.*

*Proof.* The three negative diagonal entries of $\widehat{D}$ give $q = 3$, while exactly nine of its twenty-one pairwise sums are negative, so $r = 9$. Exact parity interpolation reconstructs the characteristic coefficients and all Hurwitz determinants. Sturm's theorem gives three positive simple roots of $a_7$ and nine positive simple roots of $\Delta_6$, with the ordering (55). The characteristic polynomial of $\widehat{K}$ has one simple zero and a cubic in the squared nonzero frequencies with three positive simple roots. Exact nullspace calculations show

$$z^T \widehat{D} z > 0 \quad (0 \neq z \in \ker \widehat{K})$$

and verify positive definiteness of the three-dimensional compression of $\widehat{D}^{[2]}$ to $\ker \widehat{K}^{[2]}$ by Sylvester's criterion. Finally, rational samples in the thirteen complementary intervals give the Routh–Hurwitz word

$$U, U, U, U, U, U, U, U, U, U, U, U, S.$$

The cofactor test is even more diagnostic. In the root order (55), the first failed lower Hurwitz determinant is

| root | $\beta_1$ | $\beta_2$ | $\alpha_1$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\alpha_2$ | $\beta_7$ | $\beta_8$ | $\alpha_3$ | $\beta_9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| first failure | $\Delta_2$ | $\Delta_2$ | $\Delta_2$ | $\Delta_2$ | $\Delta_2$ | $\Delta_2$ | $\Delta_2$ | $\Delta_2$ | $\Delta_2$ | $\Delta_4$ | $\Delta_6$ | none |

Hence only $\beta_9$ has a Hurwitz complementary factor. The complete integer reconstruction, Sturm brackets, compression minors, interval sign table, neutral-factor identities, and exact algebraic-root sign audit are reproduced by the supplementary scripts

`verify_seven_dimensional_saturation.py`
`verify_boundary_admissibility.py`.

□

**Remark 8.6** (the remaining seven-dimensional target)**.** Proposition 8.1 realizes $(q, r, A, H) = (2, 8, 2, 8)$. Thus a transverse sixth component cannot arise by merely rearranging its ten current positive roots: at least one additional positive root of $\Delta_6$ is necessary, the second-compound inertia must increase from eight to nine, and the terminal compression must remain positive definite. Proposition 8.5 shows that the second saturated spectral target is itself nonempty. Its failure is now exact at the level of individual roots: only the final root is boundary-admissible, whereas a six-window $q = 3$ pencil requires eleven admissible roots. Proposition 2.5 shows that the $\Delta_2$ part of those tests is governed by one affine threshold, and Proposition 8.4 rules out the most direct $q = 2$ continuation of the existing seed. By Corollary 8.3, the open problem $c(7) = 5$ or 6 is reduced to one of fifteen channel words together with the remaining simultaneous Hurwitz-cofactor conditions inside the two saturated strata.

## 9. An eight-dimensional six-window seed

The same weak-extension mechanism can be applied once more. This time it adds a determinant notch to the final stable ray of the seven-dimensional five-window seed while preserving all five inherited components.

Let $D_7, K_7$ be the matrices in Proposition 8.1, put

$$w = (-126886, 6255, -2884, 16832, 49323, -27135, -13)^T,$$

and define

$$D_8 = \begin{pmatrix} D_7 & 0 \\ 0 & 10^{-12} \end{pmatrix}, \qquad K_8 = \begin{pmatrix} K_7 & 10^{-12} w \\ -10^{-12} w^T & 0 \end{pmatrix}. \tag{57}$$

Thus $D_8$ and $K_8$ again have a common exact scale $10^{-12}$. If $A_7(\varepsilon) = D_7 + \varepsilon K_7$, the block determinant formula gives

$$\det(D_8 + \varepsilon K_8) = \det A_7(\varepsilon)\left(10^{-12} + 10^{-24}\varepsilon^2 w^T A_7(\varepsilon)^{-1} w\right) \tag{58}$$

whenever $A_7(\varepsilon)$ is invertible. The scalar factor is positive on the inherited stable windows, changes sign twice farther out on the final ray, and is positive again at infinity.

Let $\rho < \eta_1 < \cdots < \eta_9$ be the ten positive roots of $\Delta_{7,8}(t)$, and let $\zeta_1 < \zeta_2$ be the two positive roots of $a_{8,8}(t)$.

**Proposition 9.1** (six windows in dimension eight)**.** *The strict stability set of $D_8 + \varepsilon K_8$ is*

$$\begin{aligned}(\sqrt{\eta_1}, \sqrt{\eta_2}) \cup (\sqrt{\eta_3}, \sqrt{\eta_4}) \cup (\sqrt{\eta_5}, \sqrt{\eta_6}) \\ \cup (\sqrt{\eta_7}, \sqrt{\eta_8}) \cup (\sqrt{\eta_9}, \sqrt{\zeta_1}) \cup (\sqrt{\zeta_2}, \infty).\end{aligned} \tag{59}$$

*All twelve positive guardian roots are simple and satisfy*

$$\begin{aligned}
\rho &\in (0.00316355, 0.00316356), & \eta_1 &\in (0.06172988, 0.06172989),\\
\eta_2 &\in (0.06504381, 0.06504382), & \eta_3 &\in (0.08180045, 0.08180046),\\
\eta_4 &\in (0.13474342, 0.13474343), & \eta_5 &\in (0.19768609, 0.19768610),\\
\eta_6 &\in (0.21771461, 0.21771462), & \eta_7 &\in (0.31482155, 0.31482156),\\
\eta_8 &\in (0.42948028, 0.42948029), & \eta_9 &\in (1.02804095, 1.02804096),\\
\zeta_1 &\in (1.99284793, 1.99284794), & \zeta_2 &\in (14.50137317, 14.50137318).
\end{aligned}$$

*The first window satisfies $\sqrt{\eta_1/\eta_2} > 3/4$. Moreover $50I + D_8$ is positive definite, $K_8$ has four distinct nonzero frequency pairs, and all four first real large-amplitude corrections are positive. The pencil realizes*

$$(q, r, A, H) = (2, 10, 2, 10),$$

*and in particular*

$$6 \le c(8) \le 8.$$

*Proof.* Using the common integer scale $10^{-12}$, exact parity interpolation of $q_t(s) = \det(sI + D_8 + \sqrt{t}K_8)$ gives coefficient degrees

$$(0, 1, 1, 2, 2, 3, 3, 4)$$

and Hurwitz-determinant degrees

$$(0, 1, 2, 4, 6, 9, 12, 16).$$

Sturm's theorem gives ten positive simple roots of $\Delta_{7,8}$ and two positive simple roots of $a_{8,8}$, with the displayed rational isolations and no other positive guardian roots. Exact Routh–Hurwitz evaluation in the thirteen complementary regions gives

$$\mathrm{U},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S}.$$

This proves (59); $\rho$ is the unique guardian root hidden in the initial unstable region. The first two isolations give $\eta_1/\eta_2 > 9/16$.

Exact eigenvalue isolation gives $q = \nu_-(D_8) = 2$ and $r = \nu_-(D_8^{[2]}) = 10$. The characteristic polynomial of $K_8$ is an even polynomial whose associated quartic in the squared frequencies has four positive simple roots. Hence the skew spectrum consists of four distinct nonzero conjugate pairs. Eventual stability makes the compression of $D_8^{[2]}$ to $\ker K_8^{[2]}$ positive semidefinite. Since $\deg_t \Delta_{7,8} = 12$ attains the rank-sensitive upper degree, that compression has full rank four; all four pair corrections are therefore strictly positive. Finally, $50I + D_8$ is strictly diagonally dominant with positive diagonal. The exact interpolation, Sturm counts, Routh signs, inertia calculation, frequency certificate, and blocker shift are reproduced by `verify_eight_dimensional_seed.py`. The upper estimate $c(8) \le 8$ is the universal guardian bound. □

**Proposition 9.2** (saturation filter for eight windows in dimension eight)**.** *Suppose an eight-dimensional eventually positive-stable symmetric–skew ray has eight strict stability components. Assume that $D$ and $D^{[2]}$ are nonsingular, $K$ has four distinct nonzero frequency pairs, and every positive root of $a_8\Delta_7$ is simple. Put*

$$q = \nu_-(D), \qquad r = \nu_-(D^{[2]}),$$

*and let $A$ and $H$ be the positive-root counts of $a_8$ and $\Delta_7$. Then necessarily*

$$\boxed{(q, r, A, H) = (4, 12, 4, 12)}. \tag{60}$$

*Moreover $D^{[2]}|_{\ker K^{[2]}}$ is positive definite. The first entrance uses two conjugate-pair roots: the first is hidden in the initial unstable region and the second is the left boundary of the first stable component. Among the seven later separators, exactly two are zero-eigenvalue separators and five are conjugate-pair separators. Thus all sixteen positive guardian roots are used, and exactly fifteen are boundary-admissible.*

*Proof.* Since $K$ is invertible, $a_8$ has degree at most four and the inertia capacity gives

$$A \le \min\{4, q, 8 - q\}.$$

The four simple skew-frequency planes give $\dim\ker K^{[2]} = 4$. Corollary 2.10 gives

$$H \le 12.$$

The seven separators after the first stable component consume at least fourteen positive guardian roots. Before that component the unstable dimension must fall from $q$ to zero. A simple zero crossing changes it by at most one and a simple conjugate-pair crossing by at most two, so the first entrance consumes at least $\lceil q/2 \rceil$ roots. Hence

$$A + H \ge 14 + \left\lceil \frac{q}{2} \right\rceil. \tag{61}$$

For $q = 1, 2, 3$ the upper bounds on $A+H$ are respectively $13, 14, 15$, while (61) requires $15, 15, 16$. For $q = 5, 6, 7$ the same capacity bound gives $A \le 3, 2, 1$, whereas the required totals are at least $17, 17, 18$. The cases $q = 0$ and $q = 8$

are incompatible with a nontrivial eventually stable ray. Thus $q = 4$, and equality must hold throughout: $A = 4$ and $H = 12$.

Write $h_2$ for the rank of the compression of $D^{[2]}$ to $\ker K^{[2]}$. The terminal estimate

$$H \le \min\left\{12 - \left\lceil \frac{4 - h_2}{2} \right\rceil, r, 28 - r - h_2\right\}$$

can equal twelve only when $h_2 = 4$ and $r = 12$. This proves (60) and positive definiteness of the terminal compression.

The first entrance uses exactly two roots. Reducing four unstable modes to zero in two simple crossings is possible only through two conjugate-pair crossings. The first remains hidden while two unstable modes persist; the second is boundary-admissible. The fourteen remaining roots form the seven separators. As in Proposition 8.2, each two-root separator must use one channel twice. The four determinant roots therefore form two zero-eigenvalue separators, while the ten remaining terminal roots form five conjugate-pair separators. No root remains unused. □

**Corollary 9.3** (finite eight-window word filter)**.** *Under the hypotheses of Proposition 9.2, write $A$ for a determinant-channel root and $H$ for a terminal-Hurwitz root. The increasing channel word has the form*

$$HHP_1P_2\cdots P_7, \qquad \{P_1,\dots,P_7\} = \{AA, AA, HH, HH, HH, HH, HH\}.$$

*The first $H$ root is hidden; the other fifteen roots must be boundary-admissible. There are exactly*

$$\binom{7}{2} = 21$$

*possible channel words, and each requires fifteen simultaneous complementary-factor Hurwitz tests.*

*Proof.* The entrance consumes two terminal roots. The separator allocation in Proposition 9.2 leaves two $AA$ pairs and five $HH$ pairs, whose positions determine the word. Corollary 2.4 gives the admissibility statement. □

**Proposition 9.4** (the saturated eight-dimensional chamber is nonempty)**.** *Let $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and set*

$$D_\star = \operatorname{diag}(-8, 20, -5, 6, -3, 4, -1, 2), \qquad K_\star = \operatorname{diag}(48J, 18J, 10J, 3J).$$

*Then*

$$(q, r, A, H) = (4, 12, 4, 12).$$

*The four roots of $a_8$ and the twelve roots of $\Delta_7$ are all positive, simple, and mutually distinct. Nevertheless*

$$\mathcal{S}(D_\star + \varepsilon K_\star) = \left(\sqrt{\frac{2}{9}}, \infty\right),$$

*so only the last of the sixteen guardian roots is boundary-admissible.*

*Proof.* For the four planes put

$$(c_i, u_i, v_i) = (12, 160, 2304), (1, 30, 324), (1, 12, 100), (1, 2, 9),$$

so $\delta_i = v_i t - u_i$. The determinant roots are

$$\frac{5}{72}, \qquad \frac{5}{54}, \qquad \frac{3}{25}, \qquad \frac{2}{9}.$$

Formula (22) gives

$$\Delta_7(t) = 12 \prod_{1 \le i < j \le 4} P_{ij}(t),$$

where, in lexicographic pair order,

$$\begin{aligned}
P_{12} &= 3920400t^2 - 434304t + 10140,\\
P_{13} &= 4857616t^2 - 606832t + 17952,\\
P_{14} &= 5267025t^2 - 693864t + 22572,\\
P_{23} &= 50176t^2 - 7216t + 240,\\
P_{24} &= 99225t^2 - 16974t + 720,\\
P_{34} &= 8281t^2 - 1602t + 72.
\end{aligned}$$

Each quadratic has two positive simple roots. Exact rational isolation gives the strict channel order

$$\begin{aligned}
H_{12}^- < H_{13}^- < H_{23}^- < H_{14}^- < A_1 < H_{34}^- < H_{14}^+ \\
< H_{13}^+ < H_{12}^+ < H_{24}^- < H_{23}^+ < A_2 < H_{24}^+ < A_3 < H_{34}^+ < A_4.
\end{aligned}$$

The supplement records disjoint intervals of width $10^{-8}$ for all sixteen roots. The symmetric inertia is visibly $q = 4$; the twelve negative pair sums give $r = 12$. The terminal compression is $\operatorname{diag}(12, 1, 1, 1) > 0$, and the four skew frequencies are distinct.

Each $2 \times 2$ block has positive trace $c_i$ and is positive stable exactly when $\delta_i(t) > 0$. Their direct sum is therefore stable only after the largest determinant threshold, $t = 2/9$. All earlier guardian roots are hidden inside the unstable region, while $A_4 = 2/9$ is the sole admissible boundary. The exact factorization, additive-compound identity, root order, inertias, and compression are reproduced by `verify_root_saturation.py`. □

**Proposition 9.5** (an irreducible saturated integer ray)**.** *Keep $D_\star$ from Proposition 9.4 and set*

$$K_\sharp = \begin{pmatrix}
0 & 480011 & 1 & 8 & 15 & 5 & 12 & 2 \\
-480011 & 0 & 14 & 4 & 11 & 1 & 8 & 15 \\
-1 & -14 & 0 & 180017 & 7 & 14 & 4 & 11 \\
-8 & -4 & -180017 & 0 & 3 & 10 & 17 & 7 \\
-15 & -11 & -7 & -3 & 0 & 100006 & 13 & 3 \\
-5 & -1 & -14 & -10 & -100006 & 0 & 9 & 16 \\
-12 & -8 & -4 & -17 & -13 & -9 & 0 & 30012 \\
-2 & -15 & -11 & -7 & -3 & -16 & -30012 & 0
\end{pmatrix}.$$

*Then $(D_\star, K_\sharp)$ is irreducible. Its determinant channel has four positive simple roots, its terminal-Hurwitz channel has twelve positive simple roots, and all sixteen roots are mutually distinct. Their exact Routh–Hurwitz region word is*

$$\mathrm{U}^{16}\mathrm{S}.$$

*Consequently the strict stability set is one final ray $(\sqrt{\theta}, \infty)$, where*

$$\frac{1}{450360034} < \theta < \frac{1}{450360033},$$

*and the first fifteen guardian roots are hidden.*

*Proof.* The diagonal entries of $D_\star$ are distinct and every off-diagonal entry of $K_\sharp$ is nonzero. The eigenbasis criterion used in Theorem 12.2 therefore proves irreducibility.

For exact certification, form the characteristic polynomial

$$\det(sI + D_\star + \sqrt{t}\, K_\sharp) = s^8 + a_1(t)s^7 + \cdots + a_8(t)$$

and its Hurwitz matrix. Fraction-free elimination gives

$$\begin{aligned} a_8(t) = {} & 67262247155067665467822611016873577358736t^4 \\ & - 339033028646480728467699495411501t^3 \\ & + 595171132087939668991366t^2 - 438228101878272t + 115200. \end{aligned}$$

Exact Sturm calculations give four positive simple roots of $a_8$, twelve positive simple roots of $\Delta_7$, and no common root. Disjoint rational isolating intervals order all sixteen roots. Evaluating the exact characteristic coefficients and Hurwitz determinants at one rational point in each complementary region yields $\mathrm{U}^{16}\mathrm{S}$; the last root is the largest root of $a_8$ and lies in the displayed interval. These calculations are reproduced by `verify_irreducible_saturation.py` and recorded in the machine-readable supplement. □

**Remark 9.6** (the eight-dimensional upper frontier)**.** The six-window seed of Proposition 9.1 realizes $(q, r, A, H) = (2, 10, 2, 10)$. It is therefore two determinant roots, two terminal roots, two negative directions, and two second-compound inertia units short of the unique transverse eight-window chamber. Proposition 9.4 shows that the chamber itself is nonempty and may carry all sixteen positive simple guardian roots while producing only one stable component. Proposition 9.5 shows that this maximal hidden-root geometry persists after the block structure is destroyed by a dense integer coupling. Thus the twenty-one channel words and fifteen simultaneous cofactor conditions in Corollary 9.3 encode a genuinely topological obstruction, not a missing spectral-capacity realization.

## 10. A nine-dimensional seven-window seed

The weak scalar-extension mechanism can be iterated once more, although the coupling direction must now be chosen to preserve the narrow fifth window

of the eight-dimensional seed while carving a new determinant gap farther out. Let $D_8, K_8$ be the matrices in Proposition 9.1, put

$$u = (-25097, 59116, -249779, -5111, 236778, 15688, -13, -2)^T,$$

and define

$$D_9 = \begin{pmatrix} D_8 & 0 \\ 0 & 10^{-12} \end{pmatrix}, \qquad K_9 = \begin{pmatrix} K_8 & 10^{-12}u \\ -10^{-12}u^T & 0 \end{pmatrix}. \tag{62}$$

The scalar Schur factor is positive on all six inherited stable windows, negative on one bounded interval beyond the inherited final entrance, and positive again at infinity.

Let $\rho < \eta_1 < \cdots < \eta_{11}$ be the twelve positive roots of $\Delta_{8,9}(t)$, and let $\zeta_1 < \zeta_2$ be the two positive roots of $a_{9,9}(t)$.

**Proposition 10.1** (seven windows in dimension nine)**.** *The strict stability set of $D_9 + \varepsilon K_9$ is*

$$\begin{aligned}(\sqrt{\eta_1}, \sqrt{\eta_2}) \cup (\sqrt{\eta_3}, \sqrt{\eta_4}) \cup (\sqrt{\eta_5}, \sqrt{\eta_6}) \\ \cup (\sqrt{\eta_7}, \sqrt{\eta_8}) \cup (\sqrt{\eta_9}, \sqrt{\eta_{10}}) \cup (\sqrt{\eta_{11}}, \sqrt{\zeta_1}) \cup (\sqrt{\zeta_2}, \infty).\end{aligned} \tag{63}$$

*All fourteen positive guardian roots are simple and satisfy*

$$\begin{aligned} \rho &\in (0.003163558, 0.003163559), & \eta_1 &\in (0.061729883, 0.061729884), \\ \eta_2 &\in (0.065043811, 0.065043812), & \eta_3 &\in (0.081800488, 0.081800489), \\ \eta_4 &\in (0.134741989, 0.134741990), & \eta_5 &\in (0.197712339, 0.197712340), \\ \eta_6 &\in (0.217682342, 0.217682343), & \eta_7 &\in (0.314829883, 0.314829884), \\ \eta_8 &\in (0.488158960, 0.488158961), & \eta_9 &\in (1.387450981, 1.387450982), \\ \eta_{10} &\in (1.682458599, 1.682458600), & \eta_{11} &\in (6.740829235, 6.740829236), \\ \zeta_1 &\in (15.311795791, 15.311795792), & \zeta_2 &\in (46.362567292, 46.362567293). \end{aligned}$$

*The first window satisfies $\sqrt{\eta_1/\eta_2} > 3/4$. Moreover $50I + D_9$ is positive definite, $K_9$ has one simple zero mode and four distinct nonzero frequency pairs, and all five first real large-amplitude corrections are positive. The pencil realizes*

$$(q, r, A, H) = (2, 12, 2, 12),$$

*and in particular*

$$7 \le c(9) \le 10.$$

*Proof.* Using the common integer scale $10^{-12}$, exact parity interpolation of $q_t(s) = \det(sI + D_9 + \sqrt{t}K_9)$ gives coefficient degrees

$$(0, 1, 1, 2, 2, 3, 3, 4, 4)$$

and Hurwitz-determinant degrees

$$(0, 1, 2, 4, 6, 9, 12, 16, 20).$$

Sturm's theorem gives twelve positive simple roots of $\Delta_{8,9}$ and two positive simple roots of $a_{9,9}$, with the displayed rational isolations and no other

positive guardian roots. Exact Routh–Hurwitz evaluation in the fifteen complementary regions gives

$$\mathrm{U},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S},\ \mathrm{U},\ \mathrm{S}.$$

This proves (63); $\rho$ is the unique guardian root hidden in the initial unstable region. The first two boundary isolations give $\eta_1/\eta_2 > 9/16$.

Since $D_9 = \mathrm{diag}(D_8, 10^{-12})$, the symmetric inertia remains $q = 2$. The new positive scalar forms negative pair sums with exactly the two negative eigenvalues of $D_8$, so $r = 10 + 2 = 12$. The characteristic polynomial of $K_9$ has one simple zero factor and an associated quartic in the squared nonzero frequencies with four positive simple roots. The leading coefficients of all Hurwitz determinants are positive, so the ray is eventually stable. The maximal degrees $\deg_t a_{9,9} = 4$ and $\deg_t \Delta_{8,9} = 16$ force full rank in the zero-mode and terminal asymptotic compressions; their semidefinite signs are therefore strict, giving one positive zero-mode correction and four positive pair corrections. Finally, $50I + D_9$ is strictly diagonally dominant with positive diagonal. All interpolation, global Sturm counts, isolating sign changes, Routh signs, inertias, frequency data, asymptotic ranks, and the blocker shift are reproduced by `verify_nine_dimensional_seed.py`. The upper estimate $c(9) \le 10$ is the universal guardian bound. □

## 11. Arbitrarily many plateau components

**Theorem 11.1** (unbounded plateau topology)**.** *For every integer $M \ge 2$, there exist a real symmetric positive-definite matrix $A_M$, a real skew-symmetric matrix $B_M$, and a one-dimensional common reducing blocker such that the blocker plateau has exactly $M$ connected components. The complementary dimension may be chosen as*

$$d_M = \begin{cases} 4, & M = 2, \\ 5, & M = 3, \\ 6, & M = 4, \\ 7, & M = 5, \\ 8, & M = 6, \\ 4M - 19, & M \ge 7. \end{cases}$$

*Thus no bound independent of dimension can control how often the spectral edge enters and leaves a fixed blocker plateau.*

*Proof.* For $M = 2$, use the gate block; for $M = 3$, use the five-dimensional seed; for $M = 4$, use the six-dimensional seed; for $M = 5$, use the seven-dimensional seed; and for $M = 6$, use the eight-dimensional seed. Suppose $M \ge 7$, put $m = M - 7$, and use the nine-dimensional seed from Proposition 10.1, whose stability set is

$$(A, B) \cup (C, D) \cup (E, F) \cup (G, H) \cup (J, L) \cup (N, P) \cup (Q, \infty),$$
$$0 < A < B < C < D < E < F < G < H < J < L < N < P < Q.$$

If $m > 0$, Lemma 5.1 allows $\mu < \mu_*$ to be chosen so close to $\mu_*$ that, writing $a = a(\mu)$, $b = b(\mu)$, and $c = c(\mu)$,

$$\frac{a}{b} < \frac{A}{B}, \qquad \left(\frac{c}{b}\right)^m < \frac{B}{A}.$$

Lemma 5.2 then cuts $m$ disjoint gaps out of the first seed window and leaves the later six seed components unchanged. For $m = 0$ no insertion is needed. The resulting stability set has $(m + 1) + 6 = M$ components and complementary dimension $9 + 4m = 4M - 19$.

Choose the blocker value $R = 50$, put $A_M|_F = RI + D_M$ on the block sum, and add a one-dimensional block $A_M|_E = R$, $B_M|_E = 0$. Propositions 6.1, 7.4, 8.1, 9.1, and 10.1, together with the diagonal notch and gate blocks, show that $RI + D_M$ is positive definite. Adjoining the transition endpoints yields a closed plateau with the same number of components. □

**Corollary 11.2** (extremal lower bound)**.** *For every $n \geq 9$,*

$$c(n) \geq 7 + \left\lfloor \frac{n-9}{4} \right\rfloor = \left\lfloor \frac{n+19}{4} \right\rfloor.$$

*In addition, $c(7) \geq 5$ and $c(8) \geq 6$. Together with Propositions 3.2, 3.4, 6.1, 7.4, 8.1, and 9.1, and 10.1, this gives*

$$c(1) = c(2) = c(3) = 1, \qquad c(4) = 2, \qquad c(5) = 3,$$
$$c(6) = 4, \qquad 5 \leq c(7) \leq 6, \qquad 6 \leq c(8) \leq 8, \qquad 7 \leq c(9) \leq 10.$$

*Proof.* Let $M = 7 + \lfloor (n-9)/4 \rfloor$. Theorem 11.1 uses complementary dimension

$$4M - 19 = 9 + 4 \left\lfloor \frac{n-9}{4} \right\rfloor \leq n.$$

Append stable scalar blocks if the inequality is strict. The intersection lemma shows that padding does not alter the stability set. The exact values through dimension six follow from the constructions and their matching upper bounds; Proposition 8.1 gives the stated dimension-seven interval, Proposition 9.1 gives the dimension-eight lower bound, and Proposition 10.1 gives the dimension-nine lower bound. □

## 12. Robustness and irreducible realizations

A pair $(D, K) \in \mathrm{Sym}(n, \mathbb{R}) \times \mathrm{Skew}(n, \mathbb{R})$ is called *irreducible* if no nonzero proper real subspace is invariant under both matrices. Since $D$ is symmetric and $K$ is skew-symmetric, every common invariant subspace is automatically a common reducing subspace.

**Lemma 12.1** (local persistence of plateau topology)**.** *Let $D_0 + \varepsilon K_0$ be eventually positive stable and assume $\alpha_-(D_0) \neq 0$. Assume that its positive stability boundary consists of finitely many amplitudes, that every boundary point is produced by a simple transverse imaginary-axis crossing, and that $K_0$ has simple complex spectrum, allowing one simple zero eigenvalue when $n$ is odd. If the first real correction along every eigenvector of $K_0$ is positive, then*

*there is a neighborhood $\mathcal{U}$ of $(D_0, K_0)$ in $\mathrm{Sym}(n,\mathbb{R}) \times \mathrm{Skew}(n,\mathbb{R})$ on which the number and ordering of strict stability components are constant.*

*Proof.* Choose $T$ beyond the last boundary amplitude. On the compact interval $[0,T]$, the characteristic polynomial and its roots depend continuously on $(D,K,\varepsilon)$ as a multiset, and therefore so does the left spectral edge $\alpha_-$. The assumption $\alpha_-(D_0) \neq 0$ excludes a stability-boundary endpoint at $\varepsilon = 0$. Away from disjoint small neighborhoods of the positive boundary points, $\alpha_-$ consequently has a uniform nonzero sign. At each boundary, a simple transverse root persists uniquely by the implicit-function theorem and retains its crossing direction. Hence the stability sign pattern on $[0,T]$ is locally constant.

Write $\delta = \varepsilon^{-1}$ and consider the analytic family $K + \delta D$. If the simple eigenvalues of $K_0$ are $i\kappa_j$ with normalized eigenvectors $v_j$—including $\kappa_j = 0$ in odd dimension—analytic perturbation theory gives, uniformly for $(D,K)$ in a sufficiently small compact neighborhood,

$$\nu_j(\delta) = i\kappa_j + \delta\langle Dv_j, v_j\rangle + O(\delta^2).$$

Multiplication by $\varepsilon$ yields

$$\lambda_j(D + \varepsilon K) = i\varepsilon\kappa_j + \langle Dv_j, v_j\rangle + O(\varepsilon^{-1}).$$

Uniformity follows because the skew spectral gaps remain bounded away from zero and $D$ remains bounded on that neighborhood. The corrections are real because $D$ is symmetric. Their strict positivity, together with simplicity of the skew spectrum, persists under small perturbations. After increasing $T$ if necessary, every nearby pencil is therefore positive stable for $\varepsilon \geq T$, and no additional component can enter from infinity. □

**Theorem 12.2** (robust irreducible realization)**.** *For every integer $M \geq 2$, let*

$$d_M = \begin{cases} 4, & M = 2, \\ 5, & M = 3, \\ 6, & M = 4, \\ 7, & M = 5, \\ 8, & M = 6, \\ 4M - 19, & M \geq 7. \end{cases}$$

*There is a nonempty open set*

$$\mathcal{V}_M \subset \mathrm{Sym}(d_M,\mathbb{R}) \times \mathrm{Skew}(d_M,\mathbb{R})$$

*such that every pencil $D + \varepsilon K$ in $\mathcal{V}_M$ is eventually positive stable and has exactly $M$ strict stability components on $\varepsilon > 0$. The set $\mathcal{V}_M$ may be chosen so that every pair in it is irreducible.*

*After adjoining a fixed one-dimensional blocker at $R = 50$, this gives a relatively open family of positive symmetric–skew pencils whose closed blocker plateau has exactly $M$ connected components and whose complementary pencil has no nontrivial common reducing decomposition.*

*Proof.* For $M = 2$, take the gate block; for $M = 3$, take the five-dimensional seed; for $M = 4$, take the six-dimensional seed; for $M = 5$, take the seven-dimensional seed; and for $M = 6$, take the eight-dimensional seed. For $M \geq 7$, take the nine-dimensional seed and $M - 7$ copies of a sufficiently narrow notch block, with scales supplied by Lemma 5.2. Choose all scale parameters away from the finite sets at which transition amplitudes coincide or two nonzero skew frequencies agree. This gives a $d_M$-dimensional gate or seed–notch pencil with exactly $M$ components.

The left spectral edge at zero is negative: this is immediate from the gate diagonal in the case $M = 2$, while $D_5$, $D_6$, $D_7$, $D_8$, and $D_9$ have negative eigenvalues and nonzero determinants in the seed constructions. Every notch diagonal also has a negative entry. Thus the endpoint hypothesis in Lemma 12.1 holds. All transition roots are simple and distinct, so at each transition exactly one block is critical. At every root of a determinant coefficient the corresponding penultimate Hurwitz determinant is nonzero, and at every root of that determinant the constant coefficient is positive. Lemma 2.21 therefore makes every transition a simple transverse imaginary-axis crossing.

We verify the large-amplitude hypothesis of Lemma 12.1 in closed form. For a four-dimensional block with diagonal part $D$ and skew part $K$ whose squared frequencies $\kappa_1^2 \neq \kappa_2^2$ are the roots of $r^2 - \frac{1}{2}\operatorname{tr}(K^T K)\, r + \det K$, the corrections $c_i = \langle Dv_i, v_i \rangle$ satisfy

$$c_1 + c_2 = \tfrac{1}{2}\operatorname{tr} D, \qquad \kappa_1^2 c_1 + \kappa_2^2 c_2 = -\tfrac{1}{2}\operatorname{tr}(DK^2),$$

because the eigenvectors of $K$ form an orthonormal basis, conjugate eigenvectors carry equal corrections, and $K^T K = -K^2$. For the gate block, $\operatorname{tr} D_G = \frac{55}{4}$ and $\operatorname{tr}(D_G K_G^2) = -811$, so the squared frequencies are $62 \mp 2\sqrt{957}$ and

$$c = \frac{55}{16} \pm \frac{83}{16\sqrt{957}},$$

both positive because $55^2 \cdot 957 > 83^2$. For the notch block the eigenvectors of $K_N$ do not depend on $\mu$, so each correction is affine in $\mu$; from $\operatorname{tr} D_N(0) = \frac{19}{4}$ and $\operatorname{tr}\big(D_N(0)K_N^2\big) = -\frac{2249}{4}$, the squared frequencies are $\frac{1}{2}(133 \pm \sqrt{16665})$ and

$$c(\mu) = \frac{19}{16} \pm \frac{1971}{16\sqrt{16665}} + \mu,$$

both positive for every $\mu \geq 0$ because $361 \cdot 16665 = 6016065 > 3884841 = 1971^2$. For the five-dimensional seed, Proposition 6.1, (43), and (44) certify one simple zero mode, two distinct nonzero skew-frequency pairs, and three positive corrections. For the six-dimensional seed, Proposition 7.4 certifies three simple nonzero frequency pairs and three positive corrections. For the seven-dimensional seed, Proposition 8.1 certifies one simple zero mode, three simple nonzero frequency pairs, and four positive corrections. For the eight-dimensional seed, Proposition 9.1 certifies four simple nonzero frequency pairs and four positive corrections. For the nine-dimensional seed, Proposition 10.1

certifies one simple zero mode, four simple nonzero frequency pairs, and five positive corrections. Replacing $K_N$ by $K_N/\lambda_j$ leaves the eigenvectors, hence the corrections, unchanged and divides the nonzero frequencies by $\lambda_j$. The resulting direct sum has simple skew spectrum after the stated generic choice of scales and satisfies every hypothesis of Lemma 12.1. Its exact $M$-component sign pattern therefore persists on an open neighborhood $\mathcal{U}_M$.

It remains to remove the block decomposition. Pairs with the following property are dense: $D$ has simple spectrum and, in an orthonormal eigenbasis of $D$, every off-diagonal entry of $K$ is nonzero. Indeed, simple-spectrum symmetric matrices are dense, and after fixing such a matrix the stated condition on $K$ excludes only finitely many coordinate hyperplanes. For such a pair, every $D$-invariant subspace is spanned by a subset of the eigenvectors, while the nonzero cross entries of $K$ prevent every nonempty proper subset from being invariant. The pair is therefore irreducible. The same conditions persist in a sufficiently small neighborhood of one such pair. Intersecting that neighborhood with $\mathcal{U}_M$ gives the required open set $\mathcal{V}_M$.

Finally, $50I + D$ is positive definite for the constructed pencil and remains so after shrinking the neighborhood. Adjoining the fixed blocker preserves the complementary plateau geometry. □

**Remark 12.3** (not a direct-sum pathology)**.** Orthogonal sums are used only to locate one pencil with the desired sign pattern. Theorem 12.2 shows that exact block structure is unnecessary: small dense couplings can destroy every common complementary reducing subspace without changing the number of plateau components. The criterion is insensitive to the ground field: a complex invariant subspace of a real symmetric matrix with simple spectrum is likewise spanned by a subset of its real eigenvectors, so the pairs in $\mathcal{V}_M$ admit no nontrivial common invariant subspace over $\mathbb{C}$ either.

## 13. A rational seven-component model

The general theorem uses a near-critical parameter chosen by continuity. The following larger example is completely rational and is included as a reproducible benchmark.

Set

$$\mu_0 = \frac{69}{5000}, \qquad (\lambda_1, \dots, \lambda_5) = \left(\frac{3}{5}, \frac{31}{50}, \frac{16}{25}, \frac{33}{50}, \frac{17}{25}\right).$$

Let $a^2$ be the positive root of

$$\begin{aligned} H_N(t) &= 270827112048339843750000t^2 \\ &\quad - 92198833588122717578125t - 6173679398954452105392, \end{aligned} \tag{64}$$

and let $b^2 < c^2$ be the roots of

$$\begin{aligned} A_N(t) &= 160000000000000000t^2 - 242861294675000000t \\ &\quad + 92157263626318371. \end{aligned} \tag{65}$$

Then the notch block at $\mu_0$ is stable on $(a,b)\cup(c,\infty)$, where

$$a \approx 0.630671, \qquad b \approx 0.869414, \qquad c \approx 0.872928.$$

**Proposition 13.1** (seven exact components)**.** *Let the complement be the direct sum of the gate block and five copies of*

$$D_N(\mu_0) + \frac{\varepsilon}{\lambda_j} K_N, \qquad 1 \le j \le 5.$$

*Shift the complement by $R = 5$ and adjoin a blocker at the same value. The closed plateau set on nonnegative amplitudes is exactly*

$$[L, \lambda_1 b] \cup [\lambda_1 c, \lambda_2 b] \cup [\lambda_2 c, \lambda_3 b] \cup [\lambda_3 c, \lambda_4 b]$$
$$\cup\, [\lambda_4 c, \lambda_5 b] \cup [\lambda_5 c, U] \cup [V, \infty]. \tag{66}$$

*In particular, this $25 \times 25$ real symmetric–skew model has seven plateau components.*

*Proof.* Exact rational root isolation for (28), (29), (64), and (65) gives

$$\begin{aligned} L < \lambda_1 b < \lambda_1 c < \lambda_2 b < \lambda_2 c < \lambda_3 b < \lambda_3 c \\ < \lambda_4 b < \lambda_4 c < \lambda_5 b < \lambda_5 c < U < V, \qquad \lambda_5 a < L. \end{aligned}$$

Lemma 3.1 then gives (66). □

**Remark 13.2** (the rational model is a robust witness)**.** The model of Proposition 13.1 itself satisfies every hypothesis of Lemma 12.1: its thirteen transition amplitudes arise from simple roots of the boundary polynomials (28), (29), (64), and (65), hence are simple transverse crossings by Lemma 2.21; its twelve skew frequencies, namely $(62 \mp 2\sqrt{957})^{1/2}$ for the gate and $\lambda_j^{-1}\big(\frac{1}{2}(133 \pm \sqrt{16665})\big)^{1/2}$ for the five scaled notch copies, are pairwise distinct, with all gaps exceeding $\frac{1}{20}$ (each squared frequency is a rational multiple of $62 \mp 2\sqrt{957}$ or $\frac{1}{2}(133 \pm \sqrt{16665})$, so distinctness reduces to finitely many exact comparisons in $\mathbb{Q}(\sqrt{957}, \sqrt{16665})$); and its large-amplitude corrections are the strictly positive values computed in the proof of Theorem 12.2, each notch correction shifted by $\mu_0$. Consequently the seven-component pattern persists on an open neighborhood of the model in $\mathrm{Sym}(24,\mathbb{R}) \times \mathrm{Skew}(24,\mathbb{R})$, and irreducible pencils with the same pattern exist arbitrarily close to it. The corrections also identify the asymptotic spectral edge exactly:

$$\lim_{\varepsilon \to \infty} \min\{\operatorname{Re} z : z \in \operatorname{spec}(C_\varepsilon)\} = \frac{19}{16} - \frac{1971}{16\sqrt{16665}} + \frac{69}{5000} = 0.2470460\ldots.$$

## 14. Certification and reproducibility

The exact statements used in the gate, notch, five-dimensional seed, and rational-witness arguments can be checked without floating-point root finding. The seed certificates are `verify_five_dimensional_seed.py` through `verify_nine_dimensional_seed.py`. For the two largest scalar extensions, `exact_polynomial_audit.py` checks the stored characteristic and Hurwitz polynomials on complete exact interpolation grids before the Sturm and

Routh certificates are applied. The file `verify_root_saturation.py` checks the all-dimensional block factorization and the compact eight-dimensional $(4, 12, 4, 12)$ representative. The file `verify_irreducible_saturation.py` independently reconstructs the dense integer perturbation, proves irreducibility from its support, isolates all sixteen guardian roots exactly, and checks the complete $\mathrm{U}^{16}\mathrm{S}$ Routh word. They evaluate exact Sturm counts and rational root isolations, reconstruct the low-dimensional pencils, prove their Routh–Hurwitz sign charts, check the usable first-window ratios, certify the large-amplitude corrections, and prove positive definiteness after the blocker shift.

The interpolation audits are complete identities rather than spot checks. For an $n$-dimensional symmetric–skew ray, the characteristic polynomial has degree $n$ in the spectral variable and degree at most $\lfloor n/2 \rfloor$ in $t = \varepsilon^2$. Agreement on the full rectangular grid of $(n+1)(\lfloor n/2 \rfloor + 1)$ exact integer nodes therefore proves the stored bivariate polynomial. A stored Hurwitz determinant of degree $d$ is then recomputed at $d+1$ exact $t$-nodes, which proves equality in $\mathbb{Z}[t]$. The script `verify_nine_dimensional_bareiss.py` repeats these matrix identities without SymPy, using a separate pure-Python fraction-free Bareiss determinant implementation.

The file `verify_exact_certificates.py` checks the critical polynomial, the gate and notch boundary polynomials, and the strict ordering of the thirteen transitions in the seven-component witness. The files `verify_symbolic.py` and `verify_numeric.py` independently reconstruct the displayed formulas and sample the predicted spectral sign patterns. The matrices, congruence matrix, and scaling parameters are supplied in `model_v0_47.json`. Running `pythonrun_all.py` in the supplement replays the full verification suite and checks its captured output snapshots. The release-level `verify_release.py` rejects unexpected files, checks every entry in `MANIFEST.sha256`, and then invokes the suite; with `--with-build` it also recompiles the manuscript three times and compares rendered pages. Exact package versions are recorded in `requirements.txt` and `ENVIRONMENT.txt`.

A theorem-to-certificate dependency map is included as `CERTIFICATE_MAP.md`. In brief, the universal bounds are tied to the guardian, inertia, and asymptotic-rank verifiers; dimensions five through nine to their seed certificates; the unresolved seven- and eight-dimensional frontiers to the saturation, boundary-admissibility, and lower-Hurwitz filters; and release integrity to the independent Bareiss audit, captured outputs, strict manifest, and clean-build render comparison.

For the rational witness, the narrowest deleted interval has width

$$\lambda_1(c - b) = 0.0021085\ldots.$$

Thus Proposition 13.1 is a qualitative open-set robustness statement, not a claim that this particular rational representative has a large numerical perturbation radius. The supplement reports both exact transition certificates and floating-point stress tests so that these two notions are not conflated.

## 15. Discussion

The plateau geometry of a single symmetric–skew pencil is finite because it is controlled by finitely many one-variable Hurwitz polynomials. The number of components nevertheless grows without bound as new complementary blocks are added. The construction separates two roles: the gate block determines the large stable chamber, while near-critical notch blocks carve out narrow instability windows without changing the eventual stable ray.

The result is finite dimensional. It does not by itself produce an advection–diffusion operator whose spectral edge has arbitrarily many plateau components. Realizing the construction inside a differential operator would require invariant or approximately invariant Fourier clusters together with control of the remaining spectrum. The robust irreducible theorem shows, however, that the finite-dimensional phenomenon does not rely on exact block diagonalization, and that no abstract monotonicity principle can follow from selfadjoint positivity plus skew-adjoint transport alone.

The surrounding literature supplies methods and analogies rather than the present topology theorem. Guardian maps detect generalized-stability boundaries [14]; the recent guardian-representation framework places the bialternate sum and the second additive compound in one Lie-algebraic picture [3]; and modern surveys and formulas for compound matrices are given in [2, 13]. Robust-control work studies segments and unidirectional rays [1, 7, 4], while canonical forms classify symmetric and skew pencils under equivalence or congruence [15]. The neutral-subspace argument behind Proposition 2.6 belongs to finite-dimensional indefinite inner-product theory [9]. Gyroscopic systems and delay equations exhibit repeated stability switches [11, 5, 6], while Lawley–Mattingly–Reed obtain arbitrarily many switching-rate phase transitions for a stochastic hybrid system [12]. To the author's knowledge, these sources do not prove an arbitrary-component theorem for real symmetric–skew rays together with exact low-dimensional maxima, open-set persistence, and irreducible realizations. The novelty claim is intentionally restricted to that combined statement and to the sharp separation between guardian-root and window complexity within this structured class.

The universal guardian bound in Proposition 2.2 is quadratic, with leading component count $n^2/8$, but is still far from sharp in higher dimension. Propositions 3.4, 6.1, and 7.4 settle dimensions four, five, and six exactly at two, three, and four components. Proposition 8.1 gives five components in dimension seven, and Proposition 9.1 gives six components in dimension eight, and Proposition 10.1 gives seven in dimension nine. At the opposite endpoint, Proposition 9.2 proves that eight components in dimension eight would require the unique saturated budget $(4, 12, 4, 12)$ and one of twenty-one channel words, with fifteen boundary-admissible roots. Proposition 9.4 then realizes that exact chamber with sixteen distinct positive roots but a single final stability ray. Corollary 11.2 gives $c(n) \geq \lfloor (n + 19)/4 \rfloor$ for $n \geq 9$. The

first unresolved value is therefore narrowed to

$$5 \le c(7) \le 6.$$

Proposition 8.2 shows that any transverse attainment of the upper value must lie in one of only two fully saturated root budgets, $(2, 9, 2, 9)$ or $(3, 9, 3, 9)$, with exactly one zero separator and four pair separators after the first entrance. The present five-window seed $(2, 8, 2, 8)$ is one terminal root and one unit of second-compound inertia short of the nearest admissible target. Proposition 8.5 then shows that the full $(3, 9, 3, 9)$ target is realizable with integer data, positive asymptotic compressions, and twelve simple positive guardian roots, yet only its final root is boundary-admissible and it has one stable ray. Corollary 8.3 reduces any transverse six-window construction to fifteen channel words. The affine $\Delta_2$ barrier turns its first lower-Hurwitz condition into one threshold test, while the scalar extension theorem shows that changing only the weak skew coupling of the known seed cannot reach saturation. Any proof or construction resolving $c(7)$ must therefore control the remaining cofactor geometry, not merely the two guardian root counts. The six-dimensional extremizer realizes the fully nondegenerate saturated stratum $(h, q, r, A, H) = (3, 2, 6, 2, 6)$ from Proposition 7.2; the rank-sensitive theorem remains necessary because eventual stability alone does not exclude higher-order asymptotic rescue. Proposition 2.17 shows that the guardian degree and positive-root count are sharp in every dimension even though the corresponding pencils have only one stable component. Corollary 2.19 and Proposition 9.5 show that maximal hidden root complexity survives on open sets and after irreducibility is imposed. The gap is therefore not algebraic root capacity, nongeneric multiplicity, or block reducibility, but the simultaneous cofactor geometry that turns roots into alternating windows. More broadly, it is open whether the true growth of $c(n)$ is linear, quadratic, or intermediate.

## Data and code availability

All matrices, exact polynomial data, verification programs, captured outputs, dependency versions, and checksums used in the paper are included in the supplementary release. The one-command exact audit is described in Section 14.

## Declaration of generative AI and AI-assisted technologies

Generative-AI systems were used under the author's direction for exploratory algebra, code drafting, manuscript organization, adversarial review, language editing, and typesetting. The author selected and checked the mathematical claims, proofs, references, and verification programs, takes full responsibility for the article, and lists no AI system as an author.

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